THE RAMIFICATIONS OF THE CENTRES: QUANTISED FUNCTION ALGEBRAS AT ROOTS OF UNITY

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Abstract. This paper continues the study of quantised function algebras $\mathcal{O}_\epsilon[G]$ of a semisimple group G at an ℓth root of unity ϵ. These algebras were introduced by De Concini and Lyubashenko in 1994, and studied further by De Concini and Procesi and by Gordon, amongst others. Our main purpose here is to increase understanding of the finite dimensional factor algebras $\mathcal{O}_\epsilon[G](g)$, for $g \in G$. We determine the representation type and block structure of these factors, and (for many g) describe them up to isomorphism. A series of parallel results is obtained for the quantised Borel algebras $U_\epsilon^{\geq 0}$ and $U_\epsilon^{\leq 0}$.

1. Introduction

1.1. The first substantial study of the quantised function algebra $\mathcal{O}_\epsilon[G]$ of the simply-connected semisimple group G at the ℓth root of unity ϵ appeared in [9]. It was shown there that, in close analogy with the case of a generic parameter [23], the representation theory of $\mathcal{O}_\epsilon[G]$ is stratified by double Bruhat cells in G. More precisely, $\mathcal{O}_\epsilon[G]$ contains a central sub-Hopf algebra isomorphic to $\mathcal{O}[G]$, over which $\mathcal{O}_\epsilon[G]$ is a projective module of constant rank $\ell^{\dim G}$. (In fact as we show below in Proposition 2.2, $\mathcal{O}_\epsilon[G]$ is a free $\mathcal{O}[G]$-module.) Every irreducible $\mathcal{O}_\epsilon[G]$-module is annihilated by a maximal ideal $\mathfrak{m}_g$ of $\mathcal{O}[G]$ (where $g \in G$), so that the finite dimensional representation theory of $\mathcal{O}_\epsilon[G]$ can effectively be reduced to the study of the bundle of $\ell^{\dim G}$-dimensional algebras $\mathcal{O}_\epsilon[G](g) := \mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G]$, for $g \in G$. The noncommutativity of the generic algebra $\mathcal{O}_q[G]$ induces a Poisson structure on $\mathcal{O}[G]$ (as in [10, Section 11]), and this is preserved by the group T of winding automorphisms of $\mathcal{O}_\epsilon[G]$ afforded by the one-dimensional representations of $\mathcal{O}_\epsilon[G]$. (Here, T is the maximal torus in G.) It follows (see [9, Section 9]) that if g and g' are in the same T-orbit of symplectic leaves in G, then $\mathcal{O}_\epsilon[G](g) \cong \mathcal{O}_\epsilon[G](g')$ [9, 9.3]. The T-orbits of symplectic leaves have

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been determined [20]: they are the double Bruhat cells

$$X_{w_1,w_2} = B^+ w_1 B^+ \cap B^- w_2 B^-,$$

where $w_1, w_2 \in W$, the Weyl group of G, and B^+ and B^- are fixed Borel subgroups of G.

1.2. The representation theory of $\mathcal{O}_\epsilon[G](g)$ was further studied in [11], where it was shown that the irreducible $\mathcal{O}_\epsilon[G](g)$-modules are permuted transitively by the winding automorphisms arising from the one-dimensional $\mathcal{O}_\epsilon[G](1)$-modules, and the number and dimension of irreducible $\mathcal{O}_\epsilon[G](g)$-modules was calculated - see Theorem 2.3(b)(ii). The complexity of $\mathcal{O}_\epsilon[G](g)$ was determined in [17], and hence the representation type of $\mathcal{O}_\epsilon[G](g)$ was found in many, but not all, cases - see Theorem 2.9(b).

1.3. The main purpose of this paper is to continue the analysis of the algebras $\mathcal{O}_\epsilon[G](g)$, our principal results in this direction being listed below. For $w \in W$ let $\ell(w)$ (respectively $s(w)$) denote the minimal length of an expression for w as a product of simple (respectively arbitrary) reflections in W. Let $w_1, w_2 \in W$, let $g \in X_{w_1,w_2}$ and set $w = w_2^{-1}w_1$. Let N be the number of positive roots of G, let r be the rank of G, let $\varpi_1, \dots, \varpi_r$ be a set of fundamental weights, and let $\mathfrak{S}(w_1, w_2) = \{i : 1 \leq i \leq r, w_0w_1, w_0w_2 \in \mathrm{Stab}_W(\varpi_i)\}$. By a *multiply-edged Cayley graph of F* we mean the graph got from the usual Cayley graph C of a group F with respect to a distinguished set of generators X (possibly including 1_F) by assigning a positive integer m_x to each $x \in X$ and replacing each edge of C corresponding to x by m_x edges in the same direction.

- (Theorem 3.3) For g in the fully Azumaya locus (that is, for those algebras whose irreducible modules have the maximal possible dimension ℓ^N), a complete description of $\mathcal{O}_\epsilon[G](g)$ as a direct sum of matrix rings over a truncated polynomial algebra.
- (Theorem 4.5) Determination of the representation type of $\mathcal{O}_\epsilon[G](g)$ in all cases - $\mathcal{O}_\epsilon[G](g)$ has finite type if $\ell(w_1) + \ell(w_2) > 2N - 2$, and is wild otherwise.
- (Corollary 7.4) Calculation of the number of blocks of $\mathcal{O}_\epsilon[G](g)$ and the structure of its quiver: the number of blocks is $\ell^{\mathrm{card}\,\mathfrak{S}(w_1,w_2)}$ and the quiver of each block is a multiply-edged Cayley graph of a certain elementary abelian ℓ-group of order $\ell^{r-s(w)-\mathrm{card}\,\mathfrak{S}(w_1,w_2)}$.

1.4. Here are some indications of the ingredients of the proofs of the above results. This paper is a sequel to [5], in which a key result (reproduced here as Theorem 2.5) shows in the present setting that if g is a fully Azumaya point of G then $\mathcal{O}_\epsilon[G]/\mathfrak{m}_g\mathcal{O}_\epsilon[G]$ is a complete matrix ring over $Z_g := Z(\mathcal{O}_\epsilon[G])/\mathfrak{m}_g Z(\mathcal{O}_\epsilon[G])$. A particular case of the main result of [4] (Theorem 2.6 below) identifies the Azumaya points of $\mathcal{O}_\epsilon[G]$ with the smooth points of its centre. Since $Z(\mathcal{O}_\epsilon[G])$ is

known thanks to work of Enriquez [13], Z_g can be determined when g lies under a smooth point of $Z(\mathcal{O}_\epsilon[G])$, so proving Theorem 3.3. For the analysis of representation type, after the work of [17], (at least for ℓ greater than the Coxeter number h of G), only the case (*) $\ell(w_1)+\ell(w_2)=2N-2$ remained to be dealt with. In this case (and without assuming $\ell > h$) there are essentially three possibilities for the pair (w_1, w_2), two of which yield an Azumaya point which can thus be disposed of thanks to Theorem 3.3. The third (where $w_1 = w_2$ and $\ell(w_1) = N-1$) is then analysed directly, and shown always to involve the wild algebra $\mathbb{C}[X,Y]/(X^\ell, Y^\ell)$. Finally, using deformation arguments we remove the restriction to $\ell > h$ arising in [17] - the idea is that every algebra $\mathcal{O}_\epsilon[G](g')$ with $g' \in X_{w_1', w_2'}$ and $\ell(w_1') + \ell(w_2') < 2N-2$ is a degeneration of an algebra $\mathcal{O}_\epsilon[G](g)$ for $g \in X_{w_1,w_2}$ satisfying (*). Then by a result of Geiß [16] we can deduce the wildness of $\mathcal{O}_\epsilon[G](g')$ from that of $\mathcal{O}_\epsilon[G](g)$. The two key ingredients of our work on blocks and quivers are Müller's theorem and skew group algebras. The former, which was also fundamental to [5] and which is restated here as Theorem 2.8, implies that the blocks of $\mathcal{O}_\epsilon[G](g)$ are in bijection with the maximal ideals of $Z(\mathcal{O}_\epsilon[G])$ lying over $\mathfrak{m}_g$. The latter feature because the algebras $\mathcal{O}_\epsilon[G](g)$ are skew group algebras. This follows from the following, the main result of Section 5, which is independent of the rest of the paper and which may be of interest in other contexts.

- (Theorem 5.2) Let k be an algebraically closed field and let R be a finite dimensional k-algebra whose irreducible modules are permuted simply transitively by a finite abelian group G of k-algebra automorphisms of R, with char$k$ coprime to $|G|$. Then R is isomorphic to a matrix algebra over a skew group algebra $S_1 * G$ with S_1 scalar local. The blocks and quiver of R are determined by the conjugation action of G on the Jacobson radical of S_1.

1.5. The paper also includes a series of results, paralleling those in 1.3, for the quantised Borel algebras $U_\epsilon^{\leq 0}$ and $U_\epsilon^{\geq 0}$ at an ℓth root of unity ϵ. Thus, see Theorem 2.9 and Corollary 4.6 for their representation type, and Theorems 6.7 and 6.10 for their blocks and quivers. The quantised analogues of the enveloping algebras of the positive and negative Borel subalgebras $\mathfrak{b}^+$ and $\mathfrak{b}^-$ of $\mathfrak{g} = \mathrm{Lie}(G)$ are closely connected to $\mathcal{O}_\epsilon[G]$, thanks to the dual of the multiplication map $m : B^+ \times B^- \longrightarrow G$. Coupled with the Hopf self-duality of the quantised Borel algebras, this yields an algebra embedding [9, Section 6] of $\mathcal{O}_\epsilon[G]$ into $U_\epsilon^{\leq 0} \otimes U_\epsilon^{\geq 0}$. Corresponding to the central subalgebra $\mathcal{O}[G]$ of $\mathcal{O}_\epsilon[G]$ are central subalgebras $\mathcal{O}[B^+]$ and $\mathcal{O}[B^-]$ of $U_\epsilon^{\leq 0}$ and $U_\epsilon^{\geq 0}$ respectively, and the above embedding specialises to the finite dimensional factor algebras - see Proposition 2.14 for details. Despite the apparently simpler structure of the quantised enveloping algebras of the Borels as compared with $\mathcal{O}_\epsilon[G]$, our results for the former are in many cases weaker than for the latter. There are at least two reasons for this: the coincidence of Azumaya points with smooth points

is not in general valid for the centres of quantised enveloping algebras (the details are laid out in Proposition 2.7); and in general $Z(U_\epsilon^{\geq 0})$ is not known. (This situation has been rectified in [18] where, in particular, $Z(U_\epsilon^{\geq 0})$ is described. Often, but not always, $Z(U_\epsilon^{\geq 0}) = \mathcal{O}[B^-]$ - see Theorem 6.10(ii).)

1.6. The contents are arranged as follows. In Section 2 notation is fixed and earlier work is recalled in the form most useful for present purposes. We also provide a proof of the freeness of $\mathcal{O}_\epsilon[G]$ over its central subalgebra $\mathcal{O}[G]$. In Section 3 the centre of $\mathcal{O}_\epsilon[G]$ is analysed with enough care to allow the proof of Theorem 3.3, describing $\mathcal{O}_\epsilon[G](g)$ when g is fully Azumaya. In Section 4 the representation type of $\mathcal{O}_\epsilon[G](g)$ and of $U_\epsilon^{\geq 0}(b)$ is determined. Section 5 contains the interlude on skew group algebras, culminating in Theorem 5.2 and Proposition 5.3. Sections 6 and 7 are concerned with the block and quiver structure of (respectively) $U_\epsilon^{\geq 0}(b)$ and $\mathcal{O}_\epsilon[G](g)$. Both these sections contain a number of examples. The final three paragraphs of the Introduction suggest three directions in which one might hope to extend the work described here.

1.7. The results of this paper show that the bundle of algebras $\{\mathcal{O}_\epsilon[G](g) : g \in G\}$ is a partially ordered collection of successive degenerations, progressing from the semisimple artinian algebras for $g \in X_{w_0,w_0}$, the big cell, where

$$\mathcal{O}_\epsilon[G](g) \quad \cong \quad (\mathrm{Mat}_{\ell^N}(\mathbb{C}))^{\oplus(\ell^r)},$$

towards the most degenerate algebras, for $g \in X_{e,e}$, where the ℓ^r irreducible modules are one-dimensional and there is only one block. This progressive degeneration is closely tied to the Bruhat-Chevalley order on $W \times W$, (see Lemma 4.4). We exploit this perspective in analysing representation type, for example, (as outlined in (1.4)), but it seems likely that more use can be made of similar arguments. A similar philosophy applies to other classes of algebras whose representation theory exhibits a geometric stratification, such as the quantised enveloping algebras $U_\epsilon(\mathfrak{g})$ and the modular enveloping algebras $\mathcal{U}(\mathfrak{g})$, for $\mathfrak{g}$ semisimple; but the positive evidence in these cases is more meagre than for the function algebras, which - thanks to their structure as Galois coverings of $\mathcal{O}[G]$ - provide a tractable testing ground for techniques and conjectures to apply to the more difficult cases.

1.8. A second aspect where further work may prove fruitful concerns the relations between the structure of $\mathcal{O}_\epsilon[G]$ and $\mathcal{O}_q[G]$, where q is generic. The primitive ideals of the generic algebras are also stratified by the orbits X_{w_1,w_2} of the double Bruhat cells, as shown in [23, 22]. Attempts have been made to determine the (second layer) links between these primitive ideals, [3, 24], but

the results remain incomplete. The work on blocks for $\mathcal{O}_\epsilon[G]$ presented here seems to support the belief that the links between $\mathcal{O}_\epsilon[G]$-irreducibles are given by the "lifts" modulo ℓ of links between primitive ideals in the corresponding stratum in $\mathcal{O}_q[G]$. It may be that the results here can lead to the formulation of the correct conjecture for the structure of the link group in the generic case.

1.9. Since the isomorphism type of $\mathcal{O}_\epsilon[G](g)$ is determined by the T-orbit X_{w_1,w_2} of leaves to which g belongs, it is very natural to ask precisely what information regarding (w_1, w_2) suffices to determine the isomorphism type of $\mathcal{O}_\epsilon[G](g)$. It's already clear from Theorem 2.3(b)(ii) that $\ell(w_1), \ell(w_2)$ and $s(w_2^{-1}w_1)$ are needed, but Corollary 7.4 indicates that card$\mathfrak{S}(w_1, w_2)$ may be required also. And indeed this is so - we show by example in 7.5, with $G = SL_4(\mathbb{C})$, that card$\mathfrak{S}(w_1, w_2)$ isn't a function of the other invariants listed above. Thus it remains an interesting open problem to determine a "minimal" set of isomorphism invariants, in terms of Weyl group data, for the algebras $\mathcal{O}_\epsilon[G](g)$.

2. Notations and Recollections

2.1. Let $C = (a_{ij})$ be a Cartan matrix of finite type having rank r and let $(d_1, \dots, d_r) \in \mathbb{N}^r$ have coprime entries such that $(d_i a_{ij})$ is symmetric. Let $\mathfrak{g}$ be the semisimple Lie algebra over $\mathbb{C}$ defined by C and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be its triangular decomposition. Let P and Q be the weight and root lattices of $\mathfrak{g}$ and let $(\ ,\ )$ be the associated non-degenerate bilinear form. Let $\{\alpha_1, \dots, \alpha_r\}$ be a set of simple roots determined by C and let $\{\varpi_1, \dots, \varpi_r\}$ be the corresponding fundamental weights of P. We have $(\varpi_i, \alpha_j) = \delta_{ij} d_i$. Let G be the simply-connected, semisimple algebraic group over $\mathbb{C}$ associated with C. We have Borel subgroups B^+ and B^- of G such that Lie$(B^\pm) = \mathfrak{n}^\pm \oplus \mathfrak{h}$. Let $T = B^+ \cap B^-$, a maximal torus of G. The Weyl group of G (with respect to T) is $N_G(T)/T$. This can be identified with the Weyl group associated with C. The Weyl group acts on both P and Q and the form $(\ ,\ )$ is W-invariant. There is a stratification of G:

$$G = \coprod_{w_1, w_2 \in W} X_{w_1,w_2}$$

where $X_{w_1,w_2} = B^+ w_1 B^+ \cap B^- w_2 B^-$. This restricts to a stratification of B^-:

$$B^- = \coprod_{w \in W} X_{w,e}.$$

Any element $w \in W$ can be written as a product of simple reflections or as a product of (arbitrary) reflections. We let $\ell(w)$ (respectively $s(w)$) equal the minimal length of an expression for w as a product of simple (respectively arbitrary) reflections. The longest word with respect to ℓ will be denoted w_0; recall that $\ell(w_0) = N$, where $N = \dim_{\mathbb{C}}(\mathfrak{n}^+)$, the number of positive roots. The

function $s : W \longrightarrow \mathbb{N}$ is called the rank function. It coincides with the codimension of $\mathbb{Q} \otimes_{\mathbb{Z}} P^w$ in $\mathbb{Q} \otimes_{\mathbb{Z}} P$, where we write P^w to denote the elements of P fixed by w. Let h be the Coxeter number of W. This equals the order in W of the product of the simple reflections. Throughout this paper $\epsilon \in \mathbb{C}$ will be a primitive ℓth root of unity for some natural number $\ell > 1$. Let $\theta = \sum a_i \alpha_i$ be the highest root of $\mathfrak{g}$. We will always require that ℓ is good, that is ℓ is odd and prime to the integers a_i and d_i for $1 \leq i \leq r$.

2.2. Let $U_\epsilon^{\geq 0}$ (respectively $U_\epsilon^{\leq 0}$) be the non-negative (respectively non-positive) subalgebra of the (simply-connected) quantised enveloping algebra at a root of unity, ϵ, associated to C, as defined in [8]. Let $\mathcal{O}_\epsilon[G]$ be the quantised function algebra at a root of unity, ϵ, associated to C, as defined in [9]. The ring of regular functions on B^-, $\mathcal{O}[B^-]$, is a central sub-Hopf algebra of $U_\epsilon^{\geq 0}$. Moreover, $U_\epsilon^{\geq 0}$ is free as a module over $\mathcal{O}[B^-]$ of rank $\ell^{\dim B^-}$, [7, Corollary 3.3(b)]. Similarly, the ring of regular functions on G, $\mathcal{O}[G]$, is a central sub-Hopf algebra of $\mathcal{O}_\epsilon[G]$, [9, Theorem 6.4]. We'll use θ to denote the embedding of $\mathcal{O}[G]$ into $Z(\mathcal{O}_\epsilon[G])$, and for the embedding of $\mathcal{O}[B^-]$ into $Z(U_\epsilon^{\geq 0})$. In both cases we'll denote the induced map θ^* on maximal spectra by π.

Proposition. *As an $\mathcal{O}[G]$-module, $\mathcal{O}_\epsilon[G]$ is free of rank $\ell^{\dim G}$.*

Proof. Thanks to [9, Theorem 7.2] $\mathcal{O}_\epsilon[G]$ is a projective $\mathcal{O}[G]$-module of rank $\ell^{\dim G}$. By [26] the Grothendieck group of projective modules over $\mathcal{O}[G]$ is trivial, in other words

$$K_0(\mathcal{O}[G]) \cong \mathbb{Z}.$$

In particular, if P is a projective $\mathcal{O}[G]$-module whose rank is greater than the Krull dimension of G, then P is necessarily free, [27, Theorem 11.3.7]. Since $\ell > 1$ we have $\mathrm{Kdim}\mathcal{O}[G] = \dim G < \ell^{dimG} = \mathrm{rank}\mathcal{O}_\epsilon[G]$, so the proposition follows. □

2.3. Let $b \in B^-$ and $g \in G$ and let $\mathfrak{m}_b \lhd \mathcal{O}[B^-]$ and $\mathfrak{m}_g \lhd \mathcal{O}[G]$ be the maximal ideals associated to these points. We define

$$U_\epsilon^{\geq 0}(b) \equiv \frac{U_\epsilon^{\geq 0}}{\mathfrak{m}_b U_\epsilon^{\geq 0}}, \qquad \mathcal{O}_\epsilon[G](g) \equiv \frac{\mathcal{O}_\epsilon[G]}{\mathfrak{m}_g \mathcal{O}_\epsilon[G]}.$$

In view of 2.2, these algebras have $\mathbb{C}$-dimension $\ell^{\dim B^-} = \ell^{N+r}$ and $\ell^{\dim G} = \ell^{2N+r}$ respectively.

Theorem. [8, Theorem 4.4],[9, Section 9],[11, Theorem 4.4 and Proposition 4.10] *Let $b, b' \in X_{w,e}$ and $g, g' \in X_{w_1,w_2}$ for some $w, w_1, w_2 \in W$.*
(a)(i) There is an algebra isomorphism $U_\epsilon^{\geq 0}(b) \cong U_\epsilon^{\geq 0}(b')$.
(ii) There are precisely $\ell^{r-s(w)}$ simple $U_\epsilon^{\geq 0}(b)$-modules and each simple module has dimension

$\ell^{\frac{1}{2}(\ell(w)+s(w))}$.

(b)(i) There is an algebra isomorphism $\mathcal{O}_\epsilon[G](g) \cong \mathcal{O}_\epsilon[G](g')$.

(ii) There are precisely $\ell^{r-s(w_2^{-1}w_1)}$ *simple* $\mathcal{O}_\epsilon[G](g)$*-modules and each simple module has dimension* $\ell^{\frac{1}{2}(\ell(w_1)+\ell(w_2)+s(w_2^{-1}w_1))}$.

2.4. We recall the description of the centre $Z(\mathcal{O}_\epsilon[G])$ of $\mathcal{O}_\epsilon[G]$ given in [13] and [9, Appendix]. Let U_q be the quantised enveloping algebra associated with Cartan matrix C, defined over $\mathbb{C}(q)$, with q an indeterminate. For $1 \leq i \leq r$ let $L(\varpi_i)$ be the simple U_q-module of type 1 with highest weight ϖ_i. Let v_{ϖ_i} (respectively $f_{-w_0\varpi_i}$) denote the highest weight vector of $L(\varpi_i)$ (respectively $L(\varpi_i)^*$) and let $v_{-\varpi_i}$ (respectively $f_{w_0\varpi_i}$) denote the lowest weight vector of $L(-w_0\varpi_i)$ (respectively $L(-w_0\varpi_i)^*$). These are well-defined up to scalar multiplication. We define the (quantum) matrix coefficients

$$b_{\varpi_i} = c^{\varpi_i}_{f_{-w_0\varpi_i}, v_{\varpi_i}}, \qquad c_{\varpi_i} = c^{-w_0\varpi_i}_{f_{w_0\varpi_i}, v_{-\varpi_i}}.$$

These elements can (and will) be considered as elements of $\mathcal{O}_\epsilon[G]$ after specialisation of an appropriate integral form. Let Z_q be the subalgebra of $\mathcal{O}_\epsilon[G]$ generated by the elements $b_i^k c_i^{\ell-k}$ for $1 \leq i \leq r$ and $0 \leq k \leq \ell$.

Theorem (Enriquez)**.** *The centre* $Z(\mathcal{O}_\epsilon[G])$ *of* $\mathcal{O}_\epsilon[G]$ *is isomorphic to* $\mathcal{O}[G] \otimes_{\mathcal{O}[G]\cap Z_q} Z_q$.

Remarks. (1) As a $\mathbb{C}$-algebra $\mathcal{O}[G] \cap Z_q$ is generated by $\{b^\ell_{\varpi_i}, c^\ell_{\varpi_i} : 1 \leq i \leq r\}$, and is isomorphic to $\mathbb{C}[X_1, \ldots X_r, Y_1, \ldots, Y_r]$.

(2) We can identify Z_q with $\mathbb{C}[\alpha_i(k) : 1 \leq i \leq r, 0 \leq k \leq \ell]/I$ where I is generated by the elements, for $1 \leq i \leq r$,

$$\begin{cases} \alpha_i(k)\alpha_i(k') - \alpha_i(0)\alpha_i(k+k') & \text{if } k+k' \leq \ell, \\ \alpha_i(k)\alpha_i(k') - \alpha_i(\ell)\alpha_i(k+k'-\ell) & \text{if } k+k' > \ell, \end{cases}$$

and the identification maps $\alpha_i(k)$ to $b_i^k c_i^{\ell-k}$, so that $\mathcal{O}[G] \cap Z_q \hookrightarrow Z_q$ corresponds to $X_i \longmapsto \alpha_i(0)$ and $Y_i \longmapsto \alpha_i(\ell)$.

(3) $Z_q \cong \bigotimes_{i=1}^r Z_q^i$, where Z_q^i is the algebra generated by $\{\alpha_i(k) : 0 \leq k \leq \ell\}$.

(4) As an $\mathcal{O}[G]\cap Z_q$-module, Z_q is finitely generated and free: indeed each Z_q^i is free over $\mathcal{O}[G]\cap Z_q^i$ with basis $\{1\} \cup \{\alpha_i(k) : 1 \leq k \leq \ell - 1\}$.

2.5. Given any simple $\mathcal{O}_\epsilon[G]$-module V the centre of $\mathcal{O}_\epsilon[G]$ acts by scalar multiplication thanks to Schur's lemma. Thus there is an algebra map

$$\zeta_V : Z(\mathcal{O}_\epsilon[G]) \longrightarrow \mathbb{C}$$

which we call the central character of V. We define the following subset of $\mathrm{Maxspec}(Z(\mathcal{O}_\epsilon[G]))$:

$$\mathcal{A}_{\mathcal{O}_\epsilon[G]} \equiv \{\ker(\zeta_V) : V \textit{ is a simple } \mathcal{O}_\epsilon[G]\textit{-module of maximal dimension}\}.$$

This set is non-empty and open in $\mathrm{Maxspec}(Z(\mathcal{O}_\epsilon[G]))$ and is called the *Azumaya locus of* $\mathcal{O}_\epsilon[G]$. Thanks to Theorem 2.3 we have

$$\begin{aligned} \mathcal{A}_{\mathcal{O}_\epsilon[G]} = \{\ker(\zeta_V) : \quad & V \textit{ is a simple } \mathcal{O}_\epsilon[G](g)\textit{-module for } g \in X_{w_1,w_2} \\ & \textit{with } \ell(w_1) + \ell(w_2) + s(w_2^{-1}w_1) = 2N\}. \end{aligned}$$

Following [5, Section 2.5] we also define the *fully Azumaya locus* $\mathcal{F}_{\mathcal{O}_\epsilon[G]}$ of $\mathcal{O}[G]$ with respect to $\mathcal{O}_\epsilon[G]$ to consist of those maximal ideals $\mathfrak{m}_g$ of $\mathcal{O}[G]$ (or equivalently those elements g of G) such that every irreducible $\mathcal{O}_\epsilon[G](g)$-module has dimension ℓ^N. In the notation of 2.3,

$$\pi(\mathcal{A}_{\mathcal{O}_\epsilon[G]}) = \mathcal{F}_{\mathcal{O}_\epsilon[G]} = \{g \in G : g \in X_{w_1,w_2}, \textit{ with } \ell(w_1) + \ell(w_2) + s(w_2^{-1}w_1) = 2N\},$$

and similarly for $U_\epsilon^{\geq 0}$.

Theorem. [5, Corollary 2.7] *Let* $g \in \mathcal{F}_{\mathcal{O}_\epsilon[G]}$. *Then*

$$\mathcal{O}_\epsilon[G](g) \cong \mathrm{Mat}_{\ell^N}\left(\frac{Z(\mathcal{O}_\epsilon[G])}{\mathfrak{m}_g Z(\mathcal{O}_\epsilon[G])}\right).$$

2.6. There is an alternative description of $\mathcal{A}_{\mathcal{O}_\epsilon[G]}$.

Theorem. [4, Theorem C] *The Azumaya locus* $\mathcal{A}_{\mathcal{O}_\epsilon[G]}$ *of* $\mathcal{O}_\epsilon[G]$ *coincides with the smooth locus of* $\mathrm{Maxspec}(Z(\mathcal{O}_\epsilon[G]))$.

In Section 3 we will use Theorems 2.5 and 2.6 to describe the algebras $\mathcal{O}_\epsilon[G](g)$ for all g in $\mathcal{F}_{\mathcal{O}_\epsilon[G]}$.

2.7. Theorem 2.6 is not in general true for $U_\epsilon^{\geq 0}$.

Proposition. *The Azumaya locus* $\mathcal{A}_{U_\epsilon^{\geq 0}}$ *of* $U_\epsilon^{\geq 0}$ *coincides with smooth locus of* $\mathrm{Maxspec}(Z(U_\epsilon^{\geq 0}))$ *if and only if* C *is a Cartan matrix of type* A_{2n}.

Proof. Let $Z = Z(U_\epsilon^{\geq 0})$. In these circumstances [4, Theorem 3.8] states that the Azumaya locus $\mathcal{A}_{U_\epsilon^{\geq 0}}$ of $U_\epsilon^{\geq 0}$ coincides with the smooth locus of $\mathrm{Maxspec}(Z)$ if $U_\epsilon^{\geq 0}$ is Azumaya in codimension one, that is the set of points of $\mathrm{Maxspec}(Z)$ which are not annihilators of simple $U_\epsilon^{\geq 0}$-modules of maximal dimension has codimension at least two, see [4, Corollary 1.8]. Since Z is the centre of a maximal order it is integrally closed, [27, Theorem 5.1.10(b)]. In particular $\mathrm{Maxspec}(Z)$ is smooth in codimension one. This means that the converse of [4, Theorem 3.8] is also true: if $\mathcal{A}_{U_\epsilon^{\geq 0}}$ coincides

with the smooth locus of Maxspec(Z) then $U_\epsilon^{\geq 0}$ is necessarily Azumaya in codimension one. The map induced by inclusion

$$\pi : \text{Maxspec}(Z) \longrightarrow \text{Maxspec}(Z_0) = B^-,$$

is surjective with finite fibres. The simple $U_\epsilon^{\geq 0}$-modules lying over $b \in X_{w,e}$ all have dimension $\ell^{\frac{1}{2}(\ell(w)+s(w))}$ and the maximal dimension of a simple $U_\epsilon^{\geq 0}$-module is $\ell^{\frac{1}{2}(N+s(w_0))}$. By [14, Theorem 1.1] the variety $X_{w,e}$ has dimension $\ell(w)+r$, so Maxspec(Z) is stratified by pieces $\pi^{-1}(X_{w,e})$ of dimension $\ell(w)+r$ over which the representation theory is constant. Hence $U_\epsilon^{\geq 0}$ is Azumaya in codimension one if and only if $\ell(w)+s(w) = N+s(w_0)$ for all $w \in W$ such that $N-\ell(w) \leq 1$. In other words we need only check that $s(w_0 s_i) = s(w_0)+1$ for all $1 \leq i \leq r$. By [17, Lemma 7.6], however, this is equivalent to the condition $-w_0(\alpha_i) \neq \alpha_i$ for all $1 \leq i \leq r$, in other words the involution $-w_0$ has no fixed points. This happens if and only if C is a Cartan matrix of type A_{2n}. $\square$

2.8. The following result was proved in a ring-theoretic setting by Müller, [28]. A discussion of the form given here can be found in [5, Paragraph 2.10].

Theorem. *Let $b \in B^-$ and $g \in G$. The blocks of $U_\epsilon^{\geq 0}(b)$ are in natural correspondence with the maximal ideals of $Z(U_\epsilon^{\geq 0})$ lying over $\mathfrak{m}_b$. Similarly, the blocks of $\mathcal{O}_\epsilon[G](g)$ are in correspondence with the maximal ideals of $Z(\mathcal{O}_\epsilon[G])$ lying over $\mathfrak{m}_g$.*

2.9. We recall the notion of representation type of a finite dimensional algebra T:

(i) T has finite representation type if there are a finite number of mutually non-isomorphic indecomposable T-modules;

(ii) T has tame representation type if T does not have finite representation type and if, for each dimension $d > 0$, there is a finite collection of $T-\mathbb{C}[x]$-bimodules M_i which are free as right $\mathbb{C}[x]$-modules such that every indecomposable T-module of dimension d is isomorphic to $M_i \otimes_{\mathbb{C}[x]} N$ for some i and some simple $\mathbb{C}[x]$-module N;

(iii) T has wild representation type if there is a finitely generated $T-\mathbb{C}<x,y>$-bimodule M which is free as a right $\mathbb{C}<x,y>$-module such that the functor $F(N) = M \otimes_{\mathbb{C}<x,y>} N$ from the category of finite dimensional $\mathbb{C}<x,y>$-modules to the category of finite dimensional T-modules preserves indecomposability and isomorphism classes. By [12] T falls into precisely one of the above classes: we will say T is finite, tame or wild as appropriate.

Theorem. [17, Theorem 7.1] *In addition to the usual hypotheses on ℓ, assume that $\ell > h$. Let $b \in X_{w,e}$ and $g \in X_{w_1,w_2}$ for some $w, w_1, w_2 \in W$.*

(a)(i) If $\ell(w) > N-2$ then $U_\epsilon^{\geq 0}(b)$ has finite representation type.

(ii) If $\ell(w) < N-2$ then $U_\epsilon^{\geq 0}(b)$ has wild representation type.

(b)(i) If $\ell(w_1)+\ell(w_2) > 2N-2$ then $\mathcal{O}_\epsilon[G](g)$ has finite representation type.

(ii) If $\ell(w_1)+\ell(w_2) < 2N-2$ then $\mathcal{O}_\epsilon[G](g)$ has wild representation type.

In Theorem 4.5 and Corollary 4.6 we'll complete the determination of the representation type of $\mathcal{O}_\epsilon[G](g)$ and $U_\epsilon^{\geq 0}(b)$ and remove the restriction $\ell > h$.

2.10. We have a description of the algebras occurring in Theorem 2.9(a)(i).

Proposition. [17, Theorem 7.7] *Assume $\ell > h$. Let $b \in X_{w,e}$.*

(a) Assume $\ell(w) = N$. Then $w = w_0$ and there is an algebra isomorphism

$$U_\epsilon^{\geq 0}(b) \cong \bigoplus_{j=1}^{\ell^{r-s(w_0)}} \mathrm{Mat}_{\ell^{N+s(w_0)}}(\mathbb{C}).$$

(b) Assume $\ell(w) = N-1$. Then $w = w_0 s_i$ for some i. There are two cases:

(i) $w_0(\alpha_i) = -\alpha_i$ - there is an algebra isomorphism

$$U_\epsilon^{\geq 0}(b) \cong \bigoplus_{j=1}^{\ell^{r-s(w_0)}} \mathrm{Mat}_{\ell^{\frac{1}{2}(N+s(w_0)-2)}}\left(\overline{U_\epsilon^{\geq 0}}(\mathfrak{sl}_2)\right);$$

(ii) $w_0(\alpha_i) \neq -\alpha_i$ - there is an algebra isomorphism

$$U_\epsilon^{\geq 0}(b) \cong \bigoplus_{j=1}^{\ell^{r-s(w_0)-1}} \mathrm{Mat}_{\ell^{\frac{1}{2}(N+s(w_0))}}\left(\frac{\mathbb{C}[X]}{(X^\ell)}\right).$$

We will see in Lemma 4.1 that the restriction $\ell > h$ can be removed from this proposition. There is also a corresponding description for the algebras in Theorem 2.9(b)(i) given in [17, Theorem 7.4]. One can recover this from Theorem 3.3.

2.11. Let us recall the definition of the (right) winding automorphisms for $U_\epsilon^{\geq 0}(b)$ (respectively $\mathcal{O}_\epsilon[G](g)$). Given a character of $U_\epsilon^{\geq 0}$ factoring through $\overline{U_\epsilon^{\geq 0}}$,

$$\chi : U_\epsilon^{\geq 0} \longrightarrow \overline{U_\epsilon^{\geq 0}} \longrightarrow \mathbb{C},$$

we define an automorphism τ_χ of $U_\epsilon^{\geq 0}$ by

$$\tau_\chi(x) = \sum_{(x)} x_{(1)}\chi(x_{(2)}).$$

Since $Z_0 \subseteq U_\epsilon^{\geq 0}$ is a sub-Hopf algebra on which χ agrees with the augmentation ϵ, we see that τ_χ acts as the identity on Z_0. Therefore τ_χ induces an automorphism on $U_\epsilon^{\geq 0}(b)$ for any $b \in B^-$. It is straightforward to check that for any $U_\epsilon^{\geq 0}(b)$-module M the twisted module ${}^{\tau_\chi}M$ is isomorphic to

$M \otimes \mathbb{C}_\chi$. One argues similarly for $\mathcal{O}_\epsilon[G]$. The characters of $\overline{U_\epsilon^{\geq 0}}$ are parametrised by $Q_\ell = Q/\ell Q$. Namely, for any element $\mu \in Q$ we have the one dimensional representation given by

$$E_i.1 = 0 \quad \text{and} \quad K_\lambda.1 = \epsilon^{(\lambda,\mu)}.$$

These representations are different for different choices of coset representative of ℓQ in Q, and every irreducible $\overline{U_\epsilon^{\geq 0}}$-module arises in this way by Theorem 2.3(a)(ii) applied in the case $w = e$. More generally, the following theorem is proved in [11, Theorems 4.5 and 4.10].

Theorem. *(i) Let $b \in X_{w,e}$. If $\mu \in Q$ and S is a simple $U_\epsilon^{\geq 0}(b)$-module, then ${}^{\tau_\mu}S \cong S$ if and only if $(\lambda, \mu) \in \ell\mathbb{Z}$ for all $\lambda \in P^w$.*

(ii) Let $g \in X_{w_1,w_2}$. If $\mu \in Q$ and S is a simple $\mathcal{O}_\epsilon[G](g)$-module, then ${}^{\tau_\mu}S \cong S$ if and only if $(w_1(\lambda), \mu) \in \ell\mathbb{Z}$ for all $\lambda \in P^{w_2^{-1}w_1}$.

2.12. We show that a subgroup of Q_ℓ acts simply transitively on the simple $U_\epsilon^{\geq 0}(b)$-modules (respectively simple $\mathcal{O}_\epsilon[G](g)$-modules).

Lemma. *(i) The nondegenerate form*

$$P \times Q \longrightarrow \mathbb{C}(q) : (\alpha, \beta) \longmapsto q^{(\alpha,\beta)} \tag{1}$$

induces a nondegenerate form

$$P_\ell \times Q_\ell \longrightarrow \mathbb{C} : (\alpha, \beta) \longmapsto \epsilon^{(\alpha,\beta)}. \tag{2}$$

(ii) There is an elementary abelian ℓ-subgroup of Q_ℓ which acts simply transitively on the simple $U_\epsilon^{\geq 0}(b)$-modules. If ℓ is prime to the order of w the subgroup can be chosen to be $Q^w/\ell Q^w$.

(iii) There is an elementary abelian ℓ-subgroup of Q_ℓ which acts simply transitively on the simple $\mathcal{O}_\epsilon[G](g)$-modules. If ℓ is prime to the order of $w_2^{-1}w_1$ then this subgroup can be chosen to be $Q^{w_2w_1^{-1}}/\ell Q^{w_2w_1^{-1}}$.

Proof. (i) We overline to indicate images modulo ℓ in a $\mathbb{Z}$-module. Now $P_\ell = \sum_j \overline{\mathbb{Z}\varpi_i}$ and $Q_\ell = \sum_j \overline{\mathbb{Z}\alpha_i}$, with $(\overline{\alpha_i}, \overline{\varpi_j}) = \delta_{ij}d_i$, a unit in $\overline{\mathbb{Z}}$ when $i = j$. Thus $P_\ell = (Q_\ell)^*$ and $Q_\ell = (P_\ell)^*$, as claimed.

(ii) That Q_ℓ acts transitively on the simple $U_\epsilon^{\geq 0}(b)$-modules follows from [11, Theorem 4.5]. Let $b \in X_{w,e}$, and write $\mathbb{Z}'$ for $\mathbb{Z}[d_1^{-1}, \dots, d_r^{-1}]$. We write $M' = M \otimes_{\mathbb{Z}} \mathbb{Z}'$ for a $\mathbb{Z}$-module M. Since $(\alpha_i, \varpi_j) = d_i\delta_{ij}$, there is a perfect pairing

$$(\ ,\ ) : P' \times Q' \longrightarrow \mathbb{Z}'.$$

Suppose first that $P' = P_1' \oplus P_2'$. Since $Q' = \mathrm{Hom}_{\mathbb{Z}'}(P', \mathbb{Z}')$ via the perfect pairing, $Q' = P_1'^{\perp} \oplus P_2'^{\perp}$.

Let $P'^w \subseteq P'$ and note that P'/P'^w is torsion-free since $n\lambda \in P'^w$ implies that $\lambda \in P'^w$. Hence

$P' = P'^w \oplus P'_2$ for some P'_2. Thus $Q' = Q_1 \oplus Q_2$, where $Q_1 = (P'^w)^\perp$ and $Q_2 = (P'_2)^\perp$. Let $\mu \in Q$ be such that $(P^w, \mu) \subseteq \ell\mathbb{Z}$. Since the integers d_i are prime to ℓ (thanks to our continuing hypothesis on ℓ given in Paragraph 2.1) this is equivalent to $(P'^w, \mu) \subseteq \ell\mathbb{Z}'$. Write $\mu = \mu_1 + \mu_2$ with $\mu_i \in Q_i$. Then for $\lambda \in P'^w, (\lambda, \mu_2) = (\lambda, \mu) \in \ell\mathbb{Z}'$, so that $(P', \mu_2) \subseteq \ell\mathbb{Z}'$. Since the pairing is perfect, this forces $\mu_2 \in \ell Q_2$. Hence $Q_2/\ell Q_2$ operates simply transitively on the irreducible $U_\epsilon^{\geq 0}(b)$-modules by Lemma 2.11(i). Now suppose that the order of w is prime to ℓ. Let $\mathbb{Z}'' = \mathbb{Z}'[\mathrm{ord}(w)^{-1}]$. Working over $\mathbb{Z}''$, we can argue as above to find a decomposition $P'' = P''^w \oplus P''_2$ with P''_2 $< w >$-invariant. We claim that $Q''_2 \subseteq Q''^w$. This follows from the observation that $(\mu - w\mu, P''^w) = 0 = (\mu - w\mu, P''_2)$, the first equality by $< w >$-invariance of the elements of P''^w, the second by orthogonality and the $< w >$-invariance of P''_2. Thus we have a factorisation

$$Q_2/\ell Q_2 \twoheadrightarrow Q_2/Q_2 \cap \ell Q^w \hookrightarrow Q^w/\ell Q^w.$$

Since the right hand side and the left hand side both have $\ell^{r-s(w)}$ elements this completes the proof of (ii).

(iii) is proved entirely similarly. $\square$

2.13. Recall that if S is a finite dimensional algebra on which a group G acts by algebra automorphisms then we can form the skew group algebra of S by G, written $S * G$. As a left S-module this is free with basis $g \in G$. Multiplication is given by extension of the formula $s^g g = gs$, for $s \in S$ and $g \in G$ and where s^g denotes the action of g on s.

2.14. Let $m : B^+ \times B^- \longrightarrow G$ be the multiplication map. Then m is a principal T-bundle onto the open, dense subset B^+B^- of G. Let $b_1 \in X_{w_1,e}$ and $b_2 \in X_{e,w_2}$ be unipotent, (the double Bruhat cells are T-invariant, so that we can find such representatives), and let $g = m(b_2, b_1) \in X_{w_1,w_2}$.

Proposition. [17, Section 2.9] *Let $w_1, w_2 \in W$, $g \in G$, $b_1 \in B^-$ and $b_2 \in B^+$ be as above. Then there is an algebra isomorphism*

$$\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \longrightarrow U_\epsilon^{\leq 0}(b_2) \otimes U_\epsilon^{\geq 0}(b_1).$$

3. The Azumaya locus of $\mathcal{O}_\epsilon[G]$

3.1. Recall $\mathcal{O}[G]$ and Z_q, the central subalgebras of $\mathcal{O}_\epsilon[G]$ introduced in 2.3 and 2.4 respectively. We will denote $\mathrm{Spec} Z_q$ by X and $\mathrm{Spec}\mathcal{O}[G] \cap Z_q$ by U. By Remark 2.4(1) U is isomorphic to $\mathbb{A}^{2r}$.

Lemma. *Let $\pi : X \longrightarrow U$ be the (surjective) morphism induced by the inclusion $\mathcal{O}[G] \cap Z_q \longrightarrow Z_q$. Write the points of U as $2r$-tuples $(b_1, \dots, b_r, c_1, \dots c_r)$ under the identification in Remark 2.4(1). If $p \in U$ is such that $b_i = 0 = c_i$ for some i then every point of $\pi^{-1}(p)$ is singular in X.*

Proof. It is clear that π factorises as $\pi_1 \times \ldots \times \pi_r$ where $\pi_i : \mathrm{Spec} Z_q^i \longrightarrow \mathrm{Spec}\mathcal{O}[G] \cap Z_q^i$. It is therefore enough to prove this for the case $r = 1$. By Remark 2.4(2) we can consider X as an affine variety embedded in $\mathbb{A}^{\ell+1}$ (with co-ordinate functions $\alpha_1(k)$). Under this identification π takes $(a_0, \ldots, a_\ell)$ to (a_0, a_ℓ). Note that $\pi^{-1}((0,0)) = (0, \ldots, 0)$. Indeed the equations

$$\alpha_i(k)\alpha_i(k') = \begin{cases} \alpha_i(0)\alpha_i(k+k') & \text{if } k+k' \leq \ell, \\ \alpha_i(\ell)\alpha_i(k+k'-\ell) & \text{if} k+k' > \ell, \end{cases}$$

show that $a_k^2 = 0$ as required. Define $f_{k,k'}$ as follows:

$$f_{k,k'} = \begin{cases} \alpha_i(k)\alpha_i(k') - \alpha_i(0)\alpha_i(k+k') & \text{if } k+k' \leq \ell, \\ \alpha_i(k)\alpha_i(k') - \alpha_i(\ell)\alpha_i(k+k'-\ell) & \text{if } k+k' > \ell. \end{cases}$$

Recall that for $p = (a_0, \ldots, a_\ell) \in X$ we define

$$f^{(1)}_{k,k',p} = \sum_{j=0}^{\ell} \frac{\partial f_{k,k'}}{\partial \alpha_1(j)}(p)(\alpha_1(j) - a_j).$$

Then, by definition, the tangent space of X at p is

$$T_pX = \cap_{1 \leq k,k' \leq \ell-1}\{x \in \mathbb{A}^{\ell+1} : f^{(1)}_{k,k',p} = 0).$$

Since $f_{k,k'}$ is homogeneous of degree two it follows that for all k, k'

$$f^{(1)}_{k,k',(0,0)} \equiv 0,$$

which implies that $\dim(T_0X) = \ell + 1$. Now Z_q is finite over $\mathcal{O}[G] \cap Z_q$, so that X has dimension 2. Therefore $(0, \ldots, 0) = \pi^{-1}((0,0))$ is a singular point. □

3.2. The following proposition allows us to ignore some unfavourable points of X.

Proposition. *Let $g \in G$ be such that $b^\ell_{\varpi_i}(g) = 0 = c^\ell_{\varpi_i}(g)$ for some i, $1 \leq i \leq r$. Then $\mathcal{O}_\epsilon[G](g)$ is not Azumaya.*

Proof. Using Theorem 2.6 it is enough to show that any maximal ideal of Z lying over $\mathfrak{m}_g$ is singular. Denote $\mathrm{Spec}(Z(\mathcal{O}_\epsilon[G]))$ by Y, the fibre product $X \times_U G$.

Claim. Let $\tilde{\pi} : Y \longrightarrow X$ be the projection map. If $x \in X$ is singular then any point of $\tilde{\pi}^{-1}(x)$ is singular in Y.

Proof of claim. For ease of notation let $R = Z_q$, $S = Z_q \cap \mathcal{O}[G]$ and $T = \mathcal{O}[G]$. Thus we are considering $R \otimes_S T$ where:

(i) as an S-module R is finitely generated and free;

(ii) the algebra S is smooth;

(iii) all algebras are affine domains. Let $\mathfrak{m}_R \triangleleft R$ be the maximal ideal corresponding to $x \in X$, and suppose M lies over $\mathfrak{m}_R$. Define $\mathfrak{m}_S = M \cap S$ and $\mathfrak{m}_T = M \cap T$, maximal ideals of S and T respectively. We first show that

$$(R \otimes_S T)_M = R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T}. \tag{3}$$

Note that $M = \mathfrak{m}_R \otimes_S T + R \otimes_S \mathfrak{m}_T$. This means in particular that if $y \in R \setminus \mathfrak{m}_R$ and $z \in T \setminus \mathfrak{m}_T$ then $y \otimes z \in R \otimes_S T \setminus M$. So we have an embedding

$$A := R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T} \longrightarrow (R \otimes_S T)_M. \tag{4}$$

Let $x \in R \otimes_S T \setminus M$. If A were local then $x \in A \setminus MA$ would be an invertible element, since MA is maximal. Therefore the map (4) would be an isomorphism, proving the claim. Thus it is enough to show that A is a local ring. Observe that by (i) the algebra $R/\mathfrak{m}_S R$ is finite dimensional. Therefore localising this at $\mathfrak{m}_R$ yields another finite dimensional algebra $R_{\mathfrak{m}_R}/\mathfrak{m}_S R_{\mathfrak{m}_R}$. Nakayama's lemma implies that $R_{\mathfrak{m}_R}$ is finite over $S_{\mathfrak{m}_S}$. Let $\overline{\mathfrak{m}} = \mathfrak{m}_T A$. Then

$$A/\overline{\mathfrak{m}} \cong R_{\mathfrak{m}_R}/\mathfrak{m}_S R_{\mathfrak{m}_R}$$

is finite dimensional, so Nakayama's lemma also implies that A is finite over $T_{\mathfrak{m}_T}$. Therefore $\overline{\mathfrak{m}}$ is contained in the Jacobson radical of A. However, since $A/\overline{\mathfrak{m}}$ is local it follows that $A/Jac(A)$ is local, as required. So we have proved (3). To complete the claim we must show that $A = R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T}$ has infinite global dimension. By hypothesis $R_{\mathfrak{m}_R}$ has. We have a change of rings spectral sequence

$$\mathrm{Ext}^p_A(k_A, \mathrm{Ext}^q_{R_{\mathfrak{m}_R}}(A, M)) \Longrightarrow \mathrm{Ext}^{p+q}_{R_{\mathfrak{m}_R}}(k_{\mathfrak{m}_R}, M), \tag{5}$$

for any $R_{\mathfrak{m}_R}$-module M. By Frobenius reciprocity we have

$$\mathrm{Ext}^q_{R_{\mathfrak{m}_R}}(A, M) = \mathrm{Ext}^q_{R_{\mathfrak{m}_R}}(R_{\mathfrak{m}_R} \otimes_{S_{\mathfrak{m}_S}} T_{\mathfrak{m}_T}, M) \cong \mathrm{Ext}^q_{S_{\mathfrak{m}_S}}(T_{\mathfrak{m}_T}, M).$$

As $S_{\mathfrak{m}_S}$ is smooth there exists a natural number Q such that

$$\mathrm{Ext}^q_{S_{\mathfrak{m}_S}}(T_{\mathfrak{m}_T}, M) = 0,$$

for all $q > Q$ and all $S_{\mathfrak{m}_S}$-modules M. If A were smooth there would be a natural number P such that

$$\mathrm{Ext}^p_A(k_A, M) = 0,$$

for all $p > P$ and all A-modules M. Then by (5) we have

$$\mathrm{Ext}^n_{R_{\mathfrak{m}_R}}(k_{\mathfrak{m}_R}, M) = 0,$$

for all $n > P + Q$ and $R_{\mathfrak{m}_R}$-modules M, contradicting the singularity of $R_{\mathfrak{m}_R}$. Thus A must be singular, proving the claim. The proposition now follows from Lemma 3.1 combined with the claim. $\square$

3.3. Now we can describe the algebras lying over the fully Azumaya locus - that is, we describe $\mathcal{O}_\epsilon[G](g)$ for $g \in \mathcal{F}_{\mathcal{O}_\epsilon[G]}$, in the notation of 2.6.

Theorem. *Suppose $g \in X_{w_1,w_2} \cap \mathcal{F}_{\mathcal{O}_\epsilon[G]}$; that is, $\ell(w_1)+\ell(w_2)+s(w_2^{-1}w_1) = 2N$. Let $s = s(w_2^{-1}w_1)$. Then there is an algebra isomorphism*

$$\mathcal{O}_\epsilon[G](g) \cong \bigoplus_1^{\ell^{r-s}} \mathrm{Mat}_{\ell^N}\left(\frac{k[X_1, \dots, X_s]}{(X_1^\ell, \dots, X_s^\ell)}\right).$$

Proof. Let's write Z_g for $\frac{Z(\mathcal{O}_\epsilon[G])}{\mathfrak{m}_g Z(\mathcal{O}_\epsilon[G])}$. The equivalence in the first sentence is a consequence of Theorem 2.3(b)(ii). By Theorem 2.5 there is an algebra isomorphism

$$\mathcal{O}_\epsilon[G](g) \cong \mathrm{Mat}_{\ell^N}(Z_g).$$

We have isomorphisms

$$\begin{aligned} Z_g &\cong \frac{Z_q \otimes_{Z_0 \cap Z_q} Z_0}{\mathfrak{m}_g (Z_q \otimes_{Z_0 \cap Z_q} Z_0)} \\ &\cong Z_q \otimes_{Z_0 \cap Z_q} k_{\mathfrak{m}_g} \\ &\cong \frac{Z_q}{(\mathfrak{m}_g \cap Z_q) Z_q}. \end{aligned}$$

Here $\mathfrak{m}_g \cap Z_q \lhd Z_0 \cap Z_q$ is, in the notation of Lemma 3.1, specified by $b^\ell_{\varpi_i}(g) = b_i$ and $c^\ell_{\varpi_i}(g) = c_i$. Recalling our decomposition in Remark 2.4(3),

$$Z_q = \bigotimes_{i=1}^r Z_q^i,$$

we see that Z_g is the tensor product of rings R_i, for $1 \le i \le r$, where

$$R_i := \frac{Z_q^i}{(\alpha_i(0) - b_i, \alpha_i(\ell) - c_i) Z_q^i}. \tag{6}$$

Let's describe the possible structure of the rings R_i. Since $\mathcal{O}_\epsilon[G](g)$ is Azumaya it follows from Proposition 3.2 that we never have $b_i = 0 = c_i$. There are only two cases to consider.

(i) $b_i \neq 0 \neq c_i$: in this case we have an algebra isomorphism

$$R_i \cong \frac{\mathbb{C}[X_i]}{(X_i^\ell - 1)} \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}. \tag{7}$$

Indeed sending $X_i \longmapsto \alpha_i(1)$ produces an isomorphism

$$\frac{\mathbb{C}[X_i]}{(X_i^\ell - b_i^{\ell-1}c_i)} \cong R_i.$$

Since this is a semisimple algebra of dimension ℓ, the isomorphisms in (7) are clear.

(ii) $b_i \neq 0 = c_i$ (the case $b_i = 0 \neq c_i$ is the same by symmetry): in this case we have an algebra isomorphism

$$R_i \cong \frac{\mathbb{C}[X_i]}{(X_i^\ell)}. \tag{8}$$

Again sending $X_i \longmapsto \alpha_i(1)$ yields the required isomorphism. To complete the theorem, recall that $\mathcal{O}_\epsilon[G](g)$ has exactly ℓ^{r-s} simple modules. But R_i has exactly ℓ simple modules in case (i) and a unique simple module in case (ii). Therefore

$$\begin{aligned}\frac{Z(\mathcal{O}_\epsilon[G])}{\mathfrak{m}_g Z(\mathcal{O}_\epsilon[G])} \cong \bigotimes_{i=1}^{r} R_i \quad &\cong \quad \bigotimes_{i=1}^{s} \frac{\mathbb{C}[X_i]}{(X_i^\ell)} \otimes \bigotimes_{i=s+1}^{r} \frac{\mathbb{C}[X_i]}{(X_i^\ell - 1)} \\ &\cong \quad \bigoplus_{1}^{\ell^{r-s}} \frac{\mathbb{C}[X_1, \dots , X_s]}{(X_1^\ell, \dots , X_s^\ell)}.\end{aligned}$$

□

Remark. 1. When $\mathcal{O}_\epsilon[G](g)$ is Azumaya then, by Theorem 3.3, the complexity of $\mathcal{O}_\epsilon[G](g)$ equals $2N - \ell(w_1) - \ell(w_2)$, as shown in [17].

2. By [5, Theorem 2.8], the maximal ideals $\mathfrak{m}_g$ of $\mathcal{O}[G]$ which are unramified in $Z(\mathcal{O}_\epsilon[G])$ form a (proper) subset of those for which $\mathcal{O}_\epsilon[G](g)$ is Azumaya, and one can read off at once from Theorem 3.3 that this set consists of those $\mathfrak{m}_g$ with g in X_{w_0,w_0}.

4. Representation type

4.1. To obtain general results in this section we use Theorem 3.3 to remove the restriction on ℓ in Proposition 2.10.

Lemma. *The statement of Proposition 2.10 is valid without the restriction $\ell > h$.*

Proof. For $b \in X_{w_0,e}$ this follows from Theorem (a)(ii). For $b \in X_{w,e}$ with $\ell(w) = N - 1$ the only point in [17] where the bound $\ell > h$ was required was to deduce that the algebra $U_\epsilon^{\geq 0}(b)$ is Nakayama, that is its projective indecomposable modules are uniserial. If we knew this to be so without the bound then the lemma would follow. Let $b \in X_{w,e}$ and $b' \in X_{e,w_0}$ be unipotent and let $g = b'b \in X_{w,w_0}$. By Proposition 2.14 there is an isomorphism

$$\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \longrightarrow U_\epsilon^{\leq 0}(b') \otimes U_\epsilon^{\geq 0}(b).$$

As $s(w_0 w) = 1$ the algebra $\mathcal{O}_\epsilon[G](g)$ is, by Theorem 3.3, a truncated polynomial ring in one variable. In particular it is a Nakayama algebra. By [31, Theorems 1.1 and 1.3(g)] a skew group extension over $\mathbb{C}$ of a Nakayama algebra is again Nakayama. Therefore $U_\epsilon^{\leq 0}(b') \otimes U_\epsilon^{\geq 0}(b)$ is a Nakayama algebra. By definition $b' \in X_{e,w_0}$ so by the first sentence of this proof $U_\epsilon^{\leq 0}(b')$ is a semisimple algebra, implying that the tensor product $U_\epsilon^{\leq 0}(b') \otimes U_\epsilon^{\geq 0}(b)$ is a direct product of matrix algebras with coefficients in $U_\epsilon^{\geq 0}(b)$. Hence $U_\epsilon^{\geq 0}(b)$ is Morita equivalent to a Nakayama algebra and so must be a Nakayama algebra itself. This proves the lemma. □

4.2. We require a general lemma from the theory of finite dimensional algebras.

Lemma. *Let S be a finite dimensional algebra over $\mathbb{C}$. Let G be a finite abelian group acting by automorphisms on S. Then S and the skew group algebra $S * G$ have the same representation type.*

Proof. Suppose we have an inclusion of algebras $S \subseteq T$. Suppose further that T has a S-bimodule decomposition $T = S \oplus M$. Then, by [2, Proposition 2], the representation type of S is a lower bound for the representation type of T (where finite is less than tame is less than wild). It's clear that S is a bimodule direct summand of $S * G$. The character group of G, say H, acts naturally on $S * G$ by

$$\chi(sg) = \chi(g)sg,$$

and by [31, Corollary 5.2] the algebras S and $(S * G) * H$ are Morita equivalent (this uses the fact that $|G|$ is invertible in $\mathbb{C}$). Combining this with the previous paragraph yields the lemma. □

4.3. The following lemma is the key to determining the representation type of the algebras $\mathcal{O}_\epsilon[G](g)$. Note that its validity doesn't require that $\ell > h$.

Lemma. *Let $w_1, w_2 \in W$ and suppose $g \in X_{w_1,w_2}$. If $\ell(w_1) + \ell(w_2) = 2N - 2$ then the algebra $\mathcal{O}_\epsilon[G](g)$ is wild.*

Proof. The proof is based on the fact that the truncated polynomial algebra

$$\frac{\mathbb{C}[X, Y]}{(X^\ell, Y^\ell)}$$

has wild representation type if $\ell \geq 3$. This is a consequence of [33, 1.1(c), 1.2]. In order that $\ell(w_1) + \ell(w_2) = 2N - 2$ we must have one of the following for $1 \leq i \leq r$ and $1 \leq j \leq r$:

1. $w_1 = w_0 s_i s_j, w_2 = w_0$ where $i \neq j$ (and the symmetric case obtained by exchanging the roles of w_1 and w_2);
2. $w_1 = w_0 s_i, w_2 = w_0 s_j$ where $i \neq j$;

3. $w_1 = w_0 s_i, w_2 = w_0 s_i$.

It follows from Theorem 2.3 that Cases 1 and 2 are Azumaya so, by Theorem 3.3,

$$\mathcal{O}_\epsilon[G](g) \sim_{\mathrm{Mor}} \bigoplus^{\ell^{r-2}} \frac{\mathbb{C}[X,Y]}{(X^\ell, Y^\ell)}.$$

Therefore $\mathcal{O}_\epsilon[G](g)$, and each of its blocks, has wild representation type. Suppose we are in Case 3. By Proposition 2.14 we can assume without loss of generality that there is an isomorphism

$$\gamma_g : \mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \longrightarrow U_\epsilon^{\leq 0}(b') \otimes U_\epsilon^{\geq 0}(b),$$

where

$$g = b'b,$$

for $b \in X_{w_0 s_i, e}$ and $b' \in X_{e, w_0 s_i}$ unipotent. By Lemma 4.2, it is enough to show that $U_\epsilon^{\leq 0}(b') \otimes U_\epsilon^{\geq 0}(b)$ is wild. By Proposition 2.10(b) and Lemma 4.1 the algebras $U_\epsilon^{\leq 0}(b')$ and $U_\epsilon^{\geq 0}(b)$ are isomorphic to direct sums of either

$$\mathrm{Mat}_s\left(\frac{\mathbb{C}[X]}{(X^\ell)}\right),$$

or

$$\mathrm{Mat}_t\left(\overline{U_\epsilon^{\geq 0}}(\mathfrak{sl}_2)\right).$$

Since $\mathrm{Mat}_s(A) \otimes \mathrm{Mat}_t(B) \cong \mathrm{Mat}_{st}(A \otimes B)$ it therefore suffices to show that the following algebras are wild:
(i) $\frac{\mathbb{C}[X,Y]}{(X^\ell, Y^\ell)}$;
(ii) $\overline{U_\epsilon^{\geq 0}}(\mathfrak{sl}_2) \otimes \frac{\mathbb{C}[X]}{(X^\ell)}$;
(iii) $\overline{U_\epsilon^{\geq 0}}(\mathfrak{sl}_2) \otimes \overline{U_\epsilon^{\geq 0}}(\mathfrak{sl}_2)$. It's clear, however, that the algebra in (ii) (respectively in (iii)) is a skew group ring with coefficient ring $\mathbb{C}[X,Y]/(X^\ell, Y^\ell)$ and group $\mathbb{Z}_\ell$ (respectively $\mathbb{Z}_\ell^2$). Applying Lemma 4.2 again and the comments in the first paragraph of this proof shows that these are indeed wild. $\square$

4.4. We need a couple of definitions from the theory of finite dimensional algebras, [15] and [25, Chapter II]. Let

$$\mathrm{Bil}(n) = \{\text{bilinear maps } m : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n\} \cong \mathbb{A}^{n^3},$$

and

$$\mathrm{Alg}(n) = \{\text{associative, bilinear } m \text{ which have an identity}\} \subseteq \mathrm{Bil}(n).$$

As discussed in [15] $\mathrm{Alg}(n)$ is an affine variety, locally closed in $\mathrm{Bil}(n)$. The group $GL(n)$ acts on $\mathrm{Alg}(n)$, the orbits being isomorphism classes of n dimensional algebras. We let $\mathcal{O}_A$ denote the orbit in $\mathrm{Alg}(n)$ of algebras isomorphic to A. We say that A' *is a degeneration of* A if $\mathcal{O}_{A'} \subseteq \overline{\mathcal{O}_A}$, the closure of $\mathcal{O}_A$.

Lemma. *Let $g \in X_{w_1,w_2}$ and $g' \in \overline{X_{w_1,w_2}}$. Then $\mathcal{O}_\epsilon[G](g')$ is a degeneration of $\mathcal{O}_\epsilon[G](g)$.*

Proof. By Proposition 2.2 $\mathcal{O}_\epsilon[G]$ is a free $\mathcal{O}[G]$-module of rank $t = \ell^{\dim G}$. Let $\{x_1, \dots, x_t\}$ be a basis for this module and define $c_{ij}^k \in \mathcal{O}[G]$ for $1 \leq i,j,k \leq t$ by the following equations,

$$x_i x_j = \sum_k c_{ij}^k x_k.$$

Then for any $g \in G$ the structure constants of $\mathcal{O}_\epsilon[G](g)$ with respect to the basis $\{x_i + \mathfrak{m}_g \mathcal{O}_\epsilon[G]\}$ are given by $(g(c_{ij}^k)) = (c_{ij}^k(g))$. As a result the map

$$\alpha : G \longrightarrow Alg(t) \subseteq \mathbb{A}^{t^3},$$

defined by $\alpha(g) = (c_{ij}^k(g))$, is a morphism of varieties. Let $g \in X_{w_1,w_2}$ and $g' \in \overline{X_{w_1,w_2}}$. By Theorem 2.3 X_{w_1,w_2} is a dense open set of $\overline{X_{w_1,w_2}}$ over which all algebras in the family $(\alpha(z))_{z \in \overline{X_{w_1,w_2}}}$ are isomorphic to $\mathcal{O}_\epsilon[G](g)$. It follows from [25, Proposition 3.5 and Section 3.7] that $A_{g'} \cong \mathcal{O}_\epsilon[G](g')$ is a degeneration of $\mathcal{O}_\epsilon[G](g)$. $\square$

Remark. The above proof is also valid for the reduced quantum Borels. Namely, if $b \in X_{w,e}$ and $b' \in \overline{X_{w,e}}$ then $U_\epsilon^{\geq 0}(b')$ is a degeneration of $U_\epsilon^{\geq 0}(b)$.

4.5. The following statement was proved by Geiß in [16],

(9) Let A' be a degeneration of A. If A is wild then so is A'.

This allows us to complete the classification of the representation type of the algebras $\mathcal{O}_\epsilon[G](g)$, without the restriction $\ell > h$.

Theorem. *Let ℓ be good. Let $w_1, w_2 \in W$ and suppose that $g \in X_{w_1,w_2}$. (i) If $\ell(w_1)+\ell(w_2) \geq 2N-1$ then $\mathcal{O}_\epsilon[G](g)$ has finite representation type. (ii) If $\ell(w_1) + \ell(w_2) \leq 2N-2$ then $\mathcal{O}_\epsilon[G](g)$ has wild representation type.*

Proof. The first part follows directly from Theorem 3.3. Indeed, if $\ell(w_1) + \ell(w_2) = 2N$ then the algebra $\mathcal{O}_\epsilon[G](g)$ is semisimple whilst if $\ell(w_1) + \ell(w_2) = 2N-1$ then $\mathcal{O}_\epsilon[G](g)$ is Morita equivalent to a direct sum of truncated polynomial algebras in one variable, hence Nakayama. Let $\preccurlyeq$ denote the Bruhat-Chevalley order on W. For the second part note that if $\ell(w_1) + \ell(w_2) < 2N-2$ then

there exists $u, v \in W$ such that $w_1 \preccurlyeq u$, $w_2 \preccurlyeq v$ and $\ell(u) + \ell(v) = 2N - 2$. Arguing exactly as in [32, Theorem 2.1] it follows that

$$\overline{X_{u,v}} = \overline{BuB} \cap \overline{B^- vB^-} = \coprod_{u' \preccurlyeq u, v' \preccurlyeq v} X_{u',v'},$$

so $g \in \overline{X_{u,v}}$. Let $g' \in X_{u,v}$. Then by Lemma 4.3 $\mathcal{O}_\epsilon[G](g')$ is wild and by Lemma 4.4 $\mathcal{O}_\epsilon[G](g)$ is a degeneration of $\mathcal{O}_\epsilon[G](g')$. Therefore, by (9), $\mathcal{O}_\epsilon[G](g)$ must be wild. □

4.6. We now tackle representation type for $U_\epsilon^{\geq 0}(b)$.

Corollary. *Let ℓ be good. Let $w \in W$ and suppose that $b \in X_{w,e}$. (i) If $\ell(w) \geq N - 1$ then $U_\epsilon^{\geq 0}(b)$ has finite representation type. (ii) If $\ell(w) \leq N - 2$ then $U_\epsilon^{\geq 0}(b)$ has wild representation type.*

Proof. The first part follows from Lemma 4.1 and the observation that $\overline{U_\epsilon^{\geq 0}(\mathfrak{sl}_2)}$ has finite representation type by Lemma 4.2. For (ii), arguing as in the proof of Theorem 4.5 we see that it is sufficient to show that $U_\epsilon^{\geq 0}(b)$ is wild in the case that $\ell(w) = N - 2$, that is $w = w_0 s_i s_j$ for $i \neq j$. Let $g \in X_{w_0 s_i s_j, w_0}$ be such that $g = b'b$ where $b' \in X_{e,w_0}$ and $b \in X_{w_0 s_i s_j, e}$ are unipotent. By Proposition 2.14 we have an algebra isomorphism

$$\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \cong U_\epsilon^{\leq 0}(b') \otimes U_\epsilon^{\geq 0}(b).$$

Since $b' \in X_{e,w_0}$ we have, by Proposition 2.10(a) and Lemma 4.1, an isomorphism

$$U_\epsilon^{\leq 0}(b') \cong \bigoplus^{\ell^{r-s(w_0)}} \mathrm{Mat}_{\ell^{\frac{1}{2}(N+s(w_0))}}(\mathbb{C}).$$

Therefore

$$\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r \cong \bigoplus^{\ell^{r-s(w_0)}} \mathrm{Mat}_{\ell^{\frac{1}{2}(N+s(w_0))}}\left(U_\epsilon^{\geq 0}(b)\right).$$

So $U_\epsilon^{\geq 0}(b)$ has the same representation type as $\mathcal{O}_\epsilon[G](g) * \mathbb{Z}_\ell^r$ and hence, by Lemma 4.2, as $\mathcal{O}_\epsilon[G](g)$. Now apply Theorem 4.5. □

Remark. When b is the identity element of B^- the results of the corollary were obtained in [6].

5. Algebras with group actions

5.1. Let R be a finite dimensional k-algebra and G be a finite abelian group. Assume that the characteristic of k is prime to the order of G. Suppose G acts as algebra automorphisms on R, so we have a group homomorphism

$$\tau : G \longrightarrow \mathrm{Aut}_{\text{k-alg}}(R).$$

If M is a finite dimensional R-module we let ${}^gM$ denote the R-module whose underlying abelian group is M and whose action is given by $r \cdot m = \tau(g)^{-1}(r)m$. Given $g \in G$ there is a functor

$$F_g : R - mod \longrightarrow R - mod,$$

which takes M to ${}^gM$ and sends $f : M \longrightarrow N$ to $f : {}^gM \longrightarrow {}^gN$. Fix a simple R-module V and let $V(g) = {}^gV$. Throughout this section we shall assume that

$$\{V(g) : g \in G\} \textit{ is a complete set of non-isomorphic simple } R\textit{-modules.}$$

In particular, this assumption implies that τ is a monomorphism, and that the simple modules share a fixed k-dimension, t say. Fix P, a projective cover of S, and let $P(g) = {}^gP$. By the above assumption

$$Q \cong \bigoplus_{g \in G} P(g)$$

is a projective generator for $R - mod$. Let $E = \mathrm{End}_R(Q)$. Given $g \in G$ let $\sigma_g : G \longrightarrow G$ denote the left regular action, that is $\sigma_g(h) = gh$. Considering elements of Q as ordered $|G|$-tuples of elements of P we can define $\psi_g : Q \longrightarrow Q$ to be the additive map which acts as the permutation σ_g on the $|G|$-tuple. In other words an element concentrated in the h^{th} position is sent to the gh^{th} position.

Lemma. *For $g \in G$ let $\psi_g : Q \longrightarrow Q$ be as above. Then*
(i) $\psi_g(r \cdot q) = \tau(g)(r) \cdot \psi_g(q)$ for all $r \in R$ and $q \in Q$;
(ii) $\psi_g \psi_h = \psi_{gh}$.

Proof. The second claim is obvious. For the first we can assume that q is concentrated in the h^{th} position. Then

$$\psi_g(r \cdot q) = \psi_g(\tau(h)^{-1}(r)q) = \tau(h)^{-1}(r)q,$$

where the right hand side is concentrated in the gh^{th} position. On the other hand, since $\psi_g(q)$ is non-zero only in the gh^{th} position, we have

$$\tau(g)(r) \cdot \psi_g(q) = \tau(gh)^{-1}(\tau(g)(r))q = \tau(h)^{-1}(r)q,$$

as required. $\square$

5.2. For $g \in G$ let $\tilde{\tau}(g) : E \longrightarrow E$ send ϕ to the map $\psi_g \circ \phi \circ \psi_g^{-1}$. The lemma ensures that this is a well-defined k-algebra automorphism and that the induced map

$$\tilde{\tau} : G \longrightarrow \mathrm{Aut}_{k\text{-alg}}(E^{\mathrm{op}})$$

is a group homomorphism. Repeating the comments of the first paragraph of this section we have a functor

$$\tilde{F}_g : E^{\mathrm{op}} - mod \longrightarrow E^{\mathrm{op}} - mod,$$

sending N to ${}^gN$ and fixing homomorphisms. Observe that Q is an (R, E^{op})-bimodule with $r \cdot q \cdot \phi = \phi(r \cdot q)$ for all $r \in R$, $q \in Q$ and $\phi \in E$. There is an equivalence of categories

$$R - mod \longrightarrow E^{op} - mod$$

given on objects by sending M to $\mathrm{Hom}_R(Q, M)$. The inverse equivalence sends N to $Q \otimes_{E^{\mathrm{op}}} N$. This equivalence induces two functors for each $g \in G$, namely

$$\alpha_g, \beta_g : R - mod \longrightarrow R - mod$$

where $\alpha_g(Q \otimes_{E^{\mathrm{op}}} N) = {}^g Q \otimes_{E^{\mathrm{op}}} N$ and $\beta_g(Q \otimes_{E^{\mathrm{op}}} N) = Q \otimes_{E^{\mathrm{op}}} {}^gN$. So α_g corresponds to F_g and β_g to $\tilde{F}_g$.

Proposition. *The functors α_g and β_g are naturally isomorphic.*

Proof. Let $\theta_g : Q \otimes_{E^{\mathrm{op}}} N \longrightarrow {}^{g^{-1}} Q \otimes_{E^{\mathrm{op}}} {}^gN$ send $q \otimes n$ to $\psi_g(q) \otimes n$. We must check this is well-defined. First note that, for $\phi \in E^{\mathrm{op}}, \psi_g(q\phi) = \psi_g(q)\tilde{\tau}(g)(\phi)$. Thus we have, for $q \in Q$ and $n \in N$,

$$\begin{aligned}
\theta_g(q\phi \otimes n) &= \psi_g(q\phi) \otimes n \\
&= \psi_g(q)\tilde{\tau}(g)(\phi) \otimes n \\
&= \psi_g(q) \otimes \tilde{\tau}(g)(\phi) \cdot n \\
&= \psi_g(q) \otimes \phi n \\
&= \theta_g(q \otimes \phi n).
\end{aligned}$$

Moreover θ_g is an R-module isomorphism, since, for $r \in R$,

$$\begin{aligned}\theta_g(rq \otimes n) &= \psi_g(rq) \otimes n \\ &= \tau(g)(r)\psi_g(q) \otimes n \\ &= r \cdot \psi_g(q) \otimes n \\ &= r \cdot \theta_g(q \otimes n).\end{aligned}$$

Since θ_g is natural in N it follows that $\alpha_g^{-1}\beta_g$ is naturally isomorphic to the identity functor. One shows similarly that $\beta_g^{-1}\alpha_g$ is also naturally isomorphic to the identity functor. $\square$

Given $g \in G$ let $\pi_g \in E$ be the primitive idempotent corresponding to projection onto $P(g)$ followed by the canonical injection of $P(g)$ into Q. For $h \in G$ it is easy to see that we have $\tilde{\tau}(h)(\pi_g) = \pi_{hg}$. Since ${}^{g}_{E^{\mathrm{op}}}Q \cong {}_{E^{\mathrm{op}}}Q$ for all $g \in G$, one sees that Q is a free E^{op}-module of rank t for some $t \geq 1$, so that

$$R \cong \mathrm{Mat}_t(S) \textit{ where } S \textit{ is a basic algebra on which } G \textit{ acts permuting}$$
$$\textit{a set of minimal primitive idempotents simply transitively.}$$

Notice that t as it appears in the above statement coincides with its earlier definition as the (shared) dimension of the simple R-modules. We let $\{e_g : g \in G\}$ be the above set of minimal primitive idempotents of S, and let $X = X(G)$ be the character group of G. Since G is abelian we have a decomposition

$$S = \bigoplus_{\chi \in X} S_\chi$$

where $S_\chi = \{s \in S : \tau(g)(s) = \chi(g)s\}$, so that S is an X-graded algebra. Given $\chi \in X$ we define

$$y_\chi = \sum_{g \in G} \chi^{-1}(g) e_g \in S_\chi.$$

If the exponent of G is ℓ then we find

$$y_\chi^\ell = \sum_{g \in G} \chi^{-1}(g^\ell) e_g^\ell = \sum_{g \in G} e_g = 1;$$

moreover, for $\chi, \eta \in X$ with $\chi \neq \eta$,

$$y_\chi y_\eta = y_{\chi\eta}.$$

Thus y_χ is a unit in S_χ for $\chi \in X$, and $\sum_{\chi \in X} k y_\chi$ is a subalgebra of S normalising S_1, and isomorphic to kX and hence to kG.

Theorem. *Retain the notation and hypotheses of the above paragraphs. There is an isomorphism $S \cong S_1 * G$, where the right hand side is a skew group ring. Moreover S_1 is scalar local. Thus $R \cong \mathrm{Mat}_t(S_1 * G) \cong \mathrm{Mat}_t(S_1) * G$.*

Proof. The discussion above shows that R is a skew group ring $R_1 * G$, and that we may reduce to the case where R is a basic algebra. By [29, Theorem 4.2] $J(R_1)R = J(R)$. By Lying Over for $R_1 \subseteq R$, [29, Theorem 16.6], we deduce that R_1 is a basic algebra. Thus $R_1/J(R_1)$ is a finite direct sum of copies of k. Commutativity of $R/J(R)$ forces the action of G on $R_1/J(R_1)$ to be trivial. Since there are exactly $|G|$ simple R-modules it follows that R_1 is (scalar) local. □

5.3. **Blocks and quivers.** The blocks and quiver of S, (and hence of R), are determined by the conjugation action of its subgroup G on $J(S_1)/J(S_1)^2$. We make this statement precise through the Morita equivalence of Paragraph 5.2 and the following result. See [1] for the terminology used here, recalling also the definition of a multiply-edged Cayley graph from 1.3.

Proposition. *Let T be the skew group ring $T_1 * G$ of a finite abelian group G over the scalar local finite dimensional k-algebra T_1, with k algebraically closed of characteristic not dividing $|G|$. Let J be the Jacobson radical of T_1, so J/J^2 is a kG-module under the conjugation action*

$$g.(t + J^2) \quad = \quad gtg^{-1} + J^2,$$

for $g \in G$ and $t \in J$. Let

$$J/J^2 \quad = \quad \sum_{\chi \in X(G)}^{\oplus} V_\chi^{(m_\chi)} \tag{10}$$

be the decomposition of J/J^2 as a direct sum of irreducible kG-modules under this action. Define

$$Y \quad := \quad Y(G,T) \quad = \quad < \chi : m_\chi \neq 0 > \quad \subseteq \quad X(G),$$

and

$$D \quad := \quad C(G,T) \quad = \quad C_G(J/J^2) \quad = \quad \{g \in G : \chi(g) = 1 \textit{ for all } \chi \in Y\}.$$

(i) The quiver Q_T of T has vertices $\{v_\chi : \chi \in X(G)\}$, and an arrow $v_\eta \longrightarrow v_\mu$ if and only if $\mu = \eta\chi^{-1}$ for some χ with $m_\chi \neq 0$.

(ii) The number of blocks of T is $|X(G) : Y| = |D|$.

(iii) Each block of T has an identical quiver, namely the multiply-edged Cayley graph of Y with respect to the generating set $\{\chi^{-1} : m_\chi \neq 0\}$ of Y, with m_χ copies of the edge χ^{-1} starting at each vertex .

(iv) The following statements are equivalent:

(a) The blocks of T are trivial.

(b) $Y = \{1\}$.

(c) G centralises J/J^2.

(d) T is the ordinary group ring $T_1 G$.

Proof. (i) Since $J(T) = J * G$ by [29, Theorem 4.2], $T/J(T) \cong kG$, so that T is basic and has quiver with vertices labelled by $X(G)$. By definition (see e.g.[1, page 65]), to find the arrows of the quiver of T we may assume without loss that $J^2 = 0$. The orthogonality relations for kG show that the primitive idempotents of kG (and hence of T) are $\{e_\chi = 1/|G| \sum_{g \in G} \chi(g^{-1})g : \chi \in X(G)\}$. Taking a basis for J consisting of eigenvectors $v_\chi \in V_\chi$ with respect to its structure (10) as kG-module, and noting that $e_\mu v_\chi e_\eta$ is non-zero if and only if $\mu = \eta\chi^{-1}$, one finds $e_\mu J e_\eta$ is non-zero if and only if $\mu = \eta\chi^{-1}$, proving (i).

(ii) It's clear from (i) that two vertices v_η and v_μ are in the same connected component of the quiver if and only if η and μ are in the same coset of Y in $X(G)$. The final equality is obvious, since $Y = X(G/D)$.

(iii) Immediate from (i).

(iv) (a) $\implies$ (b): By (ii).

(b) $\implies$ (c): By definition of Y.

(c) $\implies$ (d): It's easy to show that any choice of lifts to J of a basis of J/J^2 generate J as a T_1-module - see for example [1, Theorem III.1.9(a)]. Thus, if (c) holds then G operates unipotently on J and hence on T_1. The assumption on the characteristic of k ensures that the action of G on T_1 is completely reducible. So (d) follows.

(d) $\implies$ (a): Trivial. $\square$

Remark. For the applications of Proposition 5.3 below it's convenient to formulate the following easy generalisation. Namely, suppose that T is a finite dimensional k-algebra containing a group algebra kG of a group G whose order is invertible in k, such that the primitive central idempotents of T are precisely the primitive idempotents of kG. Then conclusions (i), (ii) and (iii) of the proposition remain true, with $J(T)$ replacing J.

6. Reduced quantum Borels

6.1. Let $w \in W$ and choose $b \in X_{w,e}$. From Lemma 2.12(ii) and the theory developed in Section 5, we know that $U_\epsilon^{\geq 0}(b)$ is a matrix ring over a skew group ring whose coefficient ring is scalar local. We proceed now to identify the components of this structure.

Let $w = s_{i_1} s_{i_2} \dots s_{i_t}$ be a reduced expression for w as a product of simple reflections, so

$$\ell(w) = t, \tag{11}$$

and let $\beta_{i_1}, \dots, \beta_{i_t}$ be the corresponding ordered subset of the positive roots of $\mathfrak{g}$, with corresponding PBW-type generators $E_{\beta_{i_1}}, \dots, E_{\beta_{i_t}}$ in $U_\epsilon^{\geq 0}$, [21, Theorem 8.24], which we renumber respectively as $\beta_1, \dots, \beta_t$ and as $E_{\beta_1}, \dots, E_{\beta_t}$. Writing $w_0 = ww_1$ for an element w_1 of W with $\ell(w) + \ell(w_1) = N = \ell(w_0)$, we obtain corresponding PBW-type elements $E_{\beta_1}, \dots, E_{\beta_t}, E_{\beta_{t+1}}, \dots, E_{\beta_N}$ of $U_\epsilon^{>0}$. Set $A(b)$ to be the subalgebra of $U_\epsilon^{>0}(b)$ generated by the images in the latter algebra of $E_{\beta_1}, \dots, E_{\beta_t}$. (We'll abuse notation by using the same notation for the image of E_{β_i} in $A(b)$, for $i = 1, \dots, t$, and also in $U_\epsilon^{>0}(b)$, for $i = 1, \dots, N$.) Now

$$U_\epsilon^{\geq 0}(b) = U_\epsilon^{>0}(b) * P_\ell,$$

a skew group ring. By the PBW-type theorem,

$$\dim_{\mathbb{C}}(U_\epsilon^{>0}(b)) = \ell^N,$$

with basis $\{E_{\beta_1}^{m_1} E_{\beta_2}^{m_2} \dots E_{\beta_N}^{m_N} : 0 \leq m_i < \ell\}$. In a similar way, $\dim_{\mathbb{C}}(A(b)) = \ell^t$. Note that $A(b)$ is normalised by P_ℓ in its conjugation action on $U_\epsilon^{>0}(b)$, so that $A(b) * P_\ell$ is a skew group subalgebra of $U_\epsilon^{\geq 0}(b)$, and

$$\dim_{\mathbb{C}}(A(b) * P_\ell) = \ell^{t+r}. \tag{12}$$

By [11, Theorems 4.4 and 4.5] and [17, Corollary 2.8] $A(b)$ and $A(b) * P_\ell$ are semisimple, so that

$$A(b) * P_\ell \quad \cap \quad J(U_\epsilon^{\geq 0}(b)) \quad = \quad 0. \tag{13}$$

By Theorem 2.11(i),

$$\dim_{\mathbb{C}}(U_\epsilon^{\geq 0}(b)/J(U_\epsilon^{\geq 0}(b))) \quad = \quad \ell^{r-s(w)}(\ell^{1/2(t+s(w))})^2 \quad = \quad \ell^{r+t}. \tag{14}$$

We conclude from (12), (13) and (14) that

$$U_\epsilon^{\geq 0}(b)/J(U_\epsilon^{\geq 0}(b)) \quad \cong \quad A(b) * P_\ell. \tag{15}$$

Notice that (15) explains the correspondence between simple $U_\epsilon^{\geq 0}(b)-$ and $A(b) * P_\ell-$modules, [11].

6.2. Thanks to Lemma 2.12(i) there is a canonical isomorphism between Q_ℓ and the character group of P_ℓ, given by $\alpha \longmapsto \epsilon^{(\alpha,-)}$. Thus P_ℓ and Q_ℓ are isomorphic groups, but we shall nevertheless use the two notations to denote (respectively) the group of automorphisms of $U_\epsilon^{\geq 0}(b)$ induced by conjugation by its subgroup P_ℓ of units, and the action of Q_ℓ by right winding automorphisms.

Lemma. *Keep the notation as above and in the previous subsection.*
(i) $A(b)$ acts faithfully on each simple $U_\epsilon^{\geq 0}(b)$-module.
(ii) The only ideals of $A(b)$ which are P_ℓ-invariant are 0 and $A(b)$.

Proof. (i) By Theorem 2.11(i) the winding automorphisms afforded by Q_ℓ permute the irreducible $U_\epsilon^{\geq 0}(b)$-modules transitively. Thanks to the definition of the coproduct and of the winding automorphisms (see (2.11)), the latter act trivially on $A(b)$. Hence, if a simple $A(b)$-module does not feature in a component of any one simple $U_\epsilon^{\geq 0}(b)$-module, then that simple $A(b)$-module doesn't feature in any of the simple $U_\epsilon^{\geq 0}(b)$-modules. But this would imply that $J(U_\epsilon^{\geq 0}(b)) \cap A(b) \neq 0$, contradicting (13). So each simple $U_\epsilon^{\geq 0}(b)$ -module is faithful for $A(b)$.

(ii) Suppose I were a non-zero proper P_ℓ-invariant ideal of $A(b)$. Then, thanks to (15), $\hat{I} := I * P_\ell + J(U_\epsilon^{\geq 0}(b))$ would be a proper (two-sided) ideal of $U_\epsilon^{\geq 0}(b)$, and $\hat{I}$ would then annihilate a simple $U_\epsilon^{\geq 0}(b)$-module V. Thus $IV = 0$ contradicting (i). $\square$

6.3. The following lemma is a special case of [30, Lemmas 6.1.5, 6.1.6].

Lemma. *Let R be a ring and let $1 = e_1 + e_2 + \ldots + e_n$ be a sum of orthogonal idempotents. Let G be a subgroup of the group of units of R and assume that G permutes $\{e_1, \ldots, e_n\}$ transitively by conjugation. Then $R \cong \mathrm{Mat}_n(S)$, where $S \cong e_1Re_1$.*

6.4. Now let

$$1 = e_1 + e_2 + \ldots + e_n$$

be a decomposition of $1 \in A(b)$ as a sum of primitive central idempotents of $A(b)$, and let

$$H := C_{P_\ell}(e_1) = \{x \in P_\ell : xe_1 = e_1x\}.$$

Since P_ℓ is abelian and since, by Lemma 6.2(ii),

$$P_\ell \text{ acts transitively on } \{e_1, \ldots, e_n\}, \tag{16}$$

$H = \cap_{i=1}^n C_{P_\ell}(e_i)$. Let the dimension of a simple $A(b)$-module be u, so

$$A(b) \quad = \quad (\mathrm{Mat}_u(\mathbb{C}))^{\oplus(n)} .$$

Thus $e_1(A(b) * P_\ell)e_1 \cong \mathrm{Mat}_u(\mathbb{C}) * H$, and it follows from Lemma 6.3 that

$$A(b) * P_\ell \quad \cong \quad \mathrm{Mat}_n(\mathrm{Mat}_u(\mathbb{C}) * H). \tag{17}$$

By the Skolem-Noether theorem the action of H on $\mathrm{Mat}_u(\mathbb{C})$ is by inner automorphisms. So by [29, Proposition 12.4], (17) yields

$$A(b) * P_\ell \quad = \quad \mathrm{Mat}_{nu}(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}^t\tilde{H}, \tag{18}$$

for some twisted group algebra $\mathbb{C}^t\tilde{H}$ of a group $\tilde{H}$ isomorphic to H. Thus $t : \tilde{H} \times \tilde{H} \longrightarrow \mathbb{C}^*$ is a 2-cocycle, for which we set

$$\tilde{H}_0 \quad := \quad \{h \in \tilde{H} : t(h,a) = t(a,h) \textit{ for all } a \in \tilde{H}\},$$

a subgroup of $\tilde{H}$.

Lemma. $Z(\mathrm{Mat}_{nu}(\mathbb{C}) \otimes \mathbb{C}^t\tilde{H}) = \mathbb{C}^t\tilde{H}_0$.

Proof. See [29, Ex.3, p.176]. ☐

6.5. By (18), Lemma 6.4 and the known parameters for $A(b) * P_\ell$ (see Theorem (2.11)(i)) ,

$$|\tilde{H}_0| \quad = \quad \ell^{r-s(w)}. \tag{19}$$

Tracing through the isomorphisms (17), (18) and Lemma 6.4, one sees that the centre of $A(b) * P_\ell$, $\mathbb{C}^t\tilde{H}_0$, is a group algebra of a group $\tilde{H}_0$, where

$$\tilde{H}_0 \quad = \quad \{\alpha_x x : x \in H_0\}. \tag{20}$$

Here, $\mathbb{C}^t\tilde{H}_0$ is actually an ordinary untwisted group algebra of $\tilde{H}_0$, because it is commutative, by [30, Lemma 1.2.9], and H_0 is a subgroup of H (which itself is a subgroup of P_ℓ); and, for $x \in H_0, \alpha_x \in A(b)$ is a unit, conjugation by which coincides with the action of x^{-1} by conjugation on $A(b)$. Our aim now is to identify H_0. To do so we must

(21) *assume that ℓ is prime to the order of w.*

Let

$$X := \{x \in Q_\ell : \tau_x \textit{ fixes the simple } A(b) * P_\ell - \textit{modules}\},$$

in the notation of 2.11. Thus X fixes the primitive central idempotents of $A(b) * P_\ell$. Note that Q_ℓ, (and hence X), act trivially on $A(b)$, by the definition of the coproduct. Thus, in view of (18), Lemma 6.4 and (20),

$$X \quad = \quad C_{Q_\ell}(H_0). \tag{22}$$

On the other hand Theorem 2.11(i) states that

$$X = C_{Q_\ell}(P_\ell^w). \tag{23}$$

Thus, from (22), (23) and Lemma 2.12(i) we conclude that

$$H_0 \quad = \quad C_{P_\ell}(X) \quad = \quad P_\ell^w. \tag{24}$$

We summarise what we have proved in the following theorem.

Theorem. *Let $b \in X_{w,e}$ and suppose (in addition to the standing hypotheses 2.1 on ℓ) that ℓ is prime to the order of w. Then the primitive central idempotents of $U_\epsilon^{\geq 0}(b)/J(U_\epsilon^{\geq 0}(b))$ are the images of the primitive idempotents of the subalgebra $\mathbb{C}\tilde{P}_\ell^w$ of $A(b) * P_\ell \subseteq U_\epsilon^{\geq 0}(b)$. For $x \in P_\ell^w$ there is a unit α_x in $A(b)$ such that $\tilde{P}_\ell^w = \{\alpha_x x : x \in P_\ell^w\}$.*

6.6. We require a lemma concerning roots.

Lemma. *Let $w = s_{i_1} \dots s_{i_t} \in W$ with $\ell(w) = t$, and let $A_w = \sum_{j=1}^t \mathbb{Z}\beta_j$, in the notation of 6.1. Then*

$$A_w = \sum_{j=1}^t \mathbb{Z}\alpha_{i_j}.$$

Proof. It is clear that $A_w \subseteq \sum_{j=1}^{\ell(w)} \mathbb{Z}\alpha_{i_j}$ since by construction β_j is a combination of the simple roots α_{i_k} for $1 \leq k \leq \ell(w)$. We prove the opposite inclusion by induction on $\ell(w) = t$, the case $\ell(w) \leq 1$ being trivial. Let $w' = s_{i_2} \dots s_{i_t}$ so that $\ell(w') = t - 1$, and let $\tilde{\beta}_1, \dots \tilde{\beta}_{t-1}$ be the set of positive roots corresponding to w'. We have $\beta_1 = \alpha_{i_1}$ and $\beta_k = s_{i_1}(\tilde{\beta}_{k-1}) = \tilde{\beta}_{k-1} + n_k\alpha_{i_1}$ for $2 \leq k \leq t$ and $n_k \in \mathbb{Z}$. Therefore

$$A_{w'} = \sum_{k=2}^t \mathbb{Z}\tilde{\beta}_{k-1} \subseteq A_w,$$

and so $A_{w'} + \mathbb{Z}\alpha_{i_1} \subseteq A_w$. By induction $A_{w'} = \sum_{k=2}^t \mathbb{Z}\alpha_{i_k}$ so we have $\sum_{k=1}^t \mathbb{Z}\alpha_{i_k} \subseteq A_w$ as required. □

6.7. **Blocks and quiver for $U_\epsilon^{\geq 0}(b)$.** Retain hypothesis (21). From Theorem 6.5 we see that the hypotheses of Remark 5.3 apply. To state the consequences for $U_\epsilon^{\geq 0}(b)$ it remains only to identify the groups G and D, or equivalently $X(G)$ and Y, in Proposition 5.3. We already know from Theorem 6.5 that $G = \tilde{P}_\ell^w$. Since

$$J(U_\epsilon^{\geq 0}(b)) \quad = \quad J(U_\epsilon^{>0}(b)) * P_\ell,$$

we can write

$$J_1 \quad = \quad J(U_\epsilon^{>0}(b));$$

then

$$D \quad = \quad C_{\tilde{P}_\ell^w}(J_1) \quad = \quad C_{\tilde{P}_\ell^w}(J_1/J_1^2)),$$

and we'll denote this group by C_ℓ^w.

Theorem. *Continue with the notation and hypotheses of this section. In particular, $b \in X_{w,e}$ with ℓ prime to the order of w, and w as in 6.1. Let d be the number of simple reflections not occurring in a reduced expression for w.*

(i) The number of blocks of $U_\epsilon^{\geq 0}(b)$ is $|C_\ell^w|$.

(ii) Set $B^w = (A_w)^\perp = \left(\sum_{j=1}^t \mathbb{Z}\beta_j\right)^\perp$, a subgroup of P. Then $B^w \subseteq P^w$ and $B_\ell^w := B^w/\ell B^w \subseteq \tilde{P}_\ell^w$, with $B_\ell^w \cap C_\ell^w = 0$, and $|B_\ell^w| = \ell^d$.

(iii) The quiver of each block of $U_\epsilon^{\geq 0}(b)$ is a multiply-edged Cayley graph of $\tilde{P}_\ell^w/C_\ell^w$ with respect to the generating set given by the inverses of the weights of $\tilde{P}_\ell^w$ on J_1/J_1^2. In particular, B_ℓ^w embeds in (the set of vertices of) each block.

Proof. Only (ii) and the final sentence of (iii) are not immediate from Remark 5.3. By Lemma 6.6,

$$B^w = \sum \mathbb{Z}\varpi_j \tag{25}$$

where the sum is taken over the set $\mathcal{C}_w$ of all ϖ_j such that s_j does not appear in a reduced expression of w. Thus each such ϖ_j is fixed by w and we have $B^w \subseteq P^w$. The same argument shows that $B_\ell^w \subseteq P_\ell^w$. That $|B_\ell^w| = \ell^d$ is clear from (25). Finally, note that for $x \in B_\ell^w$, the corresponding unit α_x of $A(b)$ is just a scalar, since B_ℓ^w commutes with $A(b)$. Since these scalars have trivial action on J_1/J_1^2, the final sentence of (iii) follows. □

6.8. The following lemma is useful for calculations.

Lemma. *Let $w = s_{i_1} \dots s_{i_t}$ be a reduced expression. Then $\ell(w) = s(w)$ if and only if $i_j \neq i_k$ for all $j \neq k$.*

Proof. Recall that $s(w)$ is the minimal length of w when written as a product of arbitrary reflections. Suppose that $i_j = n = i_k$. Then

$$s_{i_j} \dots s_{i_k} = (s_n s_{i_{j+1}} s_n)(s_n s_{i_{j+2}} s_n) \dots (s_n s_{i_{k-1}} s_n) = s_{s_n(\alpha_{i_{j+1}})} s_{s_n(\alpha_{i_{j+2}})} \dots s_{s_n(\alpha_{i_{k-1}})},$$

so $\ell(w) > s(w)$. Conversely suppose that $i_j \neq i_k$ for all $j \neq k$. We prove that $\ell(w) = s(w)$ by induction on $\ell(w)$, the case of $\ell(w) \leq 1$ being clear. Let $w' = s_{i_1}w$ so that $\ell(w') = \ell(w) - 1$. Suppose that $\beta = \sum n_i\alpha_i \in Q^w$. Since a reduced expression of w' does not contain the simple reflection s_{i_1} we deduce that if $w'\beta = \sum n'_i\alpha_i$ then $n_{i_1} = n'_{i_1}$. As $\beta \in Q^w$ we have

$$w'\beta = s_{i_1}\beta = \beta - < \beta, \alpha_{i_1}^\vee > \alpha_{i_1}.$$

As a result $< \beta, \alpha_{i_1}^\vee >= 0$, implying that $w'\beta = \beta$. Thus $P^w \subseteq P^{w'}$ and so we are in the case where $P^w = P^{w'} \cap P^{s_i}$, (see [11, Section 5.3]), so that $s(w) = s(w') + 1 = \ell(w') + 1 = \ell(w)$. This proves the lemma. □

6.9. We now have an upper and a lower bound on the number of blocks of $U_\epsilon^{\geq 0}(b)$ for $b \in X_{w,e}$. Namely, if k is the number of simple modules in a block then it follows from Theorem 6.7 that, with d as defined there, $\ell^d \leq k \leq \ell^{r-s(w)}$. We present a sufficient condition for these bounds to agree.

Corollary. *Let $b \in X_{w,e}$ where $w = s_{i_1} \dots s_{i_t}$ is a reduced expression such that $i_j \neq i_k$ for all $j \neq k$. Then $U_\epsilon^{\geq 0}(b)$ has a unique block.*

Proof. By Theorem 6.7(ii), under the hypothesis of the corollary B_ℓ^w has cardinality ℓ^{r-t}. By Lemma 6.8, $t = \ell(w) = s(w)$, proving the corollary, in the light of Theorem 6.7(iii). □

Remark. Under the circumstances of the corollary, its proof together with Theorem 6.7(iii) shows that the quiver of $U_\epsilon^{\geq 0}(b)$ is a multiply-edged Cayley graph of P_ℓ^w, that $P_\ell^w = \sum_{\varpi_j \in \mathcal{C}_w} \bar{\mathbb{Z}}\varpi_j$, and that the graph includes an arrow starting at each vertex for each fundamental weight ϖ_j in $\mathcal{C}_w$. But we don't know whether additional arrows can also occur in the Cayley graph, besides copies of these ones.

6.10. In many cases we can determine the number of blocks of $U_\epsilon^{\geq 0}(b)$ for all $b \in B$ very easily.

Theorem. *(i) Let $b \in X_{w,e}$ and $b' \in \overline{X_{w,e}}$. Then the number of blocks of $U_\epsilon^{\geq 0}(b')$ is no greater than the number of blocks of $U_\epsilon^{\geq 0}(b)$.*
(ii) For $b \in B$ the algebra $U_\epsilon^{\geq 0}(b)$ has at most $\ell^{r-s(w_0)}$ blocks, and this upper bound is attained for b in the (open, dense) stratum $X_{w_0,e}$.
(iii) The following are equivalent:
1. $Z(U_\epsilon^{\geq 0}) = Z_0$;
2. $s(w_0) = r$;

3. the Cartan matrix C is of type B_r, C_r, D_r (r even), E_7, E_8, F_4 or G_2;

4. for all $b \in B$, $U_\epsilon^{\geq 0}(b)$ has a unique block.

Proof. (i) By [15, Proposition 2.7] the set $\{A \in \mathrm{Alg}(n) : \text{number of blocks of } A \leq s\}$ is closed in $\mathrm{Alg}(n)$. Thus, by Remark 4.4 the algebra $U_\epsilon^{\geq 0}(b')$ has no more blocks than $U_\epsilon^{\geq 0}(b)$. (ii) For b in the non-empty open set $X_{w_0,e}$ of B^-, $U_\epsilon^{\geq 0}(b)$ is semisimple with $\ell^{r-s(w_0)}$ simple modules, by Proposition 2.10. The first part of the claim now follows from (i). (iii) $1 \iff 2$: Since Z_0 is integrally closed, $Z_0 \subsetneq Z(U_\epsilon^{\geq 0})$ if and only if these algebras have distinct quotient fields, and this happens if and only if, for a generic maximal ideal $\mathfrak{m}$ of Z_0, $Z/\mathfrak{m}Z \supsetneq \mathbb{C}$. Since the generic maximal ideal of Z_0 is contained in a maximal ideal of Z unramified over Z_0, we can conclude that $Z_0 \subsetneq Z(U_\epsilon^{\geq 0})$ if and only if there is a maximal ideal of Z_0 contained in at least two maximal ideals of Z. So the equivalence now follows from Theorem 2.8.

$2 \iff 4$: By (ii).

$2 \iff 3$: This can be read off from [17, Table 1]. $\square$

6.11. **Examples.** We illustrate the above analysis with a couple of examples. (i) $\mathrm{R} = \overline{U_\epsilon^{\geq 0}}$. In this case the simply transitive group of winding automorphisms is Q_ℓ. A simple calculation yields:

$$\tau_{\alpha_i}(E_j) = E_j$$
$$\tau_{\alpha_i}(K_\lambda) = \epsilon^{(\lambda,\alpha_i)} K_\lambda.$$

A set of primitive idempotents of R is given by

$$e_\mu = \sum_{\lambda \in P_\ell} \epsilon^{(\lambda,\mu)} K_\lambda,$$

for $\mu \in Q_\ell$. Thus $\tau_{\alpha_i}(e_\mu) = e_{\mu+\alpha_i}$. In the notation of Section 5 it is easy to check that R_1 is the subalgebra of R generated by the elements E_i for $1 \leq i \leq r$, so is just $\overline{U_\epsilon^{>0}}$. Under the identification of $X(Q_\ell)$ with P_ℓ using the inner product, it is straightforward to check that $y_\lambda = K_\lambda$ for all $\lambda \in P_\ell$. Therefore the analysis of the Section 5 recovers the well-known description of $\overline{U_\epsilon^{\geq 0}}$ as a skew group extension of $\overline{U_\epsilon^{>0}}$. Note too that in this case $B_\ell^e = P_\ell$ so there is a unique block. In fact one can check that J_1/J_1^2 has a basis given by the images of $E_1, \dots, E_r$. The quiver of R (together with its relations) is described in [19]. (ii) $R = U_\epsilon^{\geq 0}(b)$ where $b \in X_{w,e}$ with $w = w_0 s_i$. The following descriptions can be read off from [17, Theorem 7.7]. We present the algebras in the format of Proposition 5.2, that is as matrix rings over skew group algebras whose coefficient rings are scalar local. There are two cases, depending on the value of $w_0(\alpha_i)$. In the ring-theoretic context of Proposition 5.3 the dichotomy is determined by whether or not $\tilde{P}_\ell^w$ acts trivially on the Jacobson

radical.

(a) $w_0(\alpha_i) = -\alpha_i$. Here,

$$U_\epsilon^{\geq 0}(b) \quad \cong \quad \mathrm{Mat}_{\ell^{1/2(N-1+s(w))}}(\overline{U_\epsilon^{\geq 0}}(\mathfrak{sl}_2) \otimes \mathbb{C}C_\ell^w),$$

where $|C_\ell^w| = \ell^{r-s(w)-1} = \ell^{r-s(w_0)}$ and we note that $\overline{U_\epsilon^{\geq 0}}(\mathfrak{sl}_2)$ is a skew group extension of a basic algebra as in (i).

(b) $w_0(\alpha_i) \neq -\alpha_i$. In this case,

$$U_\epsilon^{\geq 0}(b) \quad \cong \quad \mathrm{Mat}_{\ell^{1/2(N-1+s(w))}}(\mathbb{C}[X]/ < X^\ell > \otimes \mathbb{C}C_\ell^w),$$

where $C_\ell^w = \tilde{P}_\ell^w$, and $|\tilde{P}_\ell^w| = \ell^{r-s(w)} = \ell^{r-s(w_0)-1}$.

7. Reduced quantised function algebras

7.1. Recall the definition of $b_{\varpi_i}, c_{\varpi_i} \in \mathcal{O}_\epsilon[G]$ given in 2.4.

Lemma. *Let $g \in G$. The number of blocks in $\mathcal{O}_\epsilon[G](g)$ equals ℓ^d where d is the cardinality of the set $\{1 \leq i \leq r : b^\ell_{\varpi_i}(g) \neq 0 \neq c^\ell_{\varpi_i}(g)\}$.*

Proof. We freely use the notation of the proof of Theorem 3.3. By Theorem 2.8 it is sufficient to show that Z_g has exactly d maximal ideals. By the proof of Theorem 3.3

$$Z_g \cong \bigotimes_{i=1}^r R_i,$$

where R_i is as in (6). Let $b_i = b^\ell_{\varpi_i}(g)$ and $c_i = c^\ell_{\varpi_i}(g)$. We have already seen in (7) that if $b_i \neq 0 \neq c_i$ then R_i is isomorphic to $\mathbb{C}^\ell$, whilst if $b_i \neq 0$ and $c_i = 0$ then R_i is isomorphic to $\mathbb{C}[X]/(X^\ell)$ by (8). It remains to consider the case $b_i = 0 = c_i$. In this case inclusion provides an isomorphism

$$\frac{\mathbb{C}[X_1, \ldots, X_{\ell-1}]}{I'_i} \longrightarrow R_i$$

where I'_i is the ideal generated by $X_k X_{k'}$ for all k and k'. Since this is manifestly a local ring, the result follows from these calculations and Theorem 2.8. $\square$

7.2. As noted in [9, Appendix] the elements $b^\ell_{\varpi_i}, c^\ell_{\varpi_i} \in \mathcal{O}[G]$ can be interpreted as the classical analogues of the quantum matrix coefficients b_{ϖ_i} and c_{ϖ_i}. In other words $b^\ell_{\varpi_i}$ (respectively $c^\ell_{\varpi_i}$) is the classical matrix coefficient $\tilde{c}^{\varpi_i}_{f_{-w_0\varpi_i}, v_{\varpi_i}}$ (respectively $\tilde{c}^{-w_0\varpi_i}_{f_{w_0\varpi_i}, v_{-\varpi_i}}$). Here we have used $\tilde{c}$ to distinguish the classical matrix coefficients from their quantum analogues.

Lemma. *Let $g \in X_{w_1,w_2}$. The algebra $\mathcal{O}_\epsilon[G](g)$ has exactly ℓ^d blocks where d is the cardinality of the set $\{1 \leq i \leq r : w_0 w_1, w_0 w_2 \in \mathrm{Stab}_W(\varpi_i)\}$.*

Proof. Let's write $B_i \in \mathcal{O}[G]$ for the regular function $b_{\varpi_i}^{\ell}$, and similarly C_i for $c_{\varpi_i}^{\ell}$. Let w be an arbitrary element of W. We have $f_{-w_0\varpi_i}(bwb'v_{\varpi_i}) \in \mathbb{C}^* f_{-w_0\varpi_i}(bwv_{\varpi_i})$ for $b, b' \in B$ since v_{ϖ_i} is a highest weight vector. Hence $B_i(bwb') = f_{-w_0\varpi_i}(bwb'v_{\varpi_i})$ is not identically zero if and only if wv_{ϖ_i} is a lowest weight vector, and in this case it is non-zero for all $b, b' \in B$. Note that this happens if and only if $w\varpi_i = w_0\varpi_i$. A similar analysis applied to C_i shows $C_i(bwb') \neq 0$ for some $b, b' \in B$ if and only if $C_i(bwb') \neq 0$ for all $b, b' \in B$ if and only if $wv_{-\varpi_i}$ is a highest weight vector. That is, if and only if $-w\varpi_i = -w_0\varpi_i$. We conclude from the previous paragraph that if $g \in X_{w_1,w_2}$ then $B_i(g) \neq 0$ and $C_i(g) \neq 0$ if and only if $w_0w_1, w_0w_2 \in \mathrm{Stab}_W(\varpi_i)$. The lemma now follows from Lemma 7.1. $\square$

7.3. Recall the definition of the winding automorphisms given in 2.11. For the rest of the paper we will fix $w_1, w_2 \in W$ and assume that ℓ is prime to $\mathrm{ord}(w_2^{-1}w_1)$.

Theorem. *Let $g \in X_{w_1,w_2}$ and let $w = w_2^{-1}w_1$. Let $N(w_1, w_2) \subseteq Q_\ell^w$ be the normaliser of one (hence any) block of $\mathcal{O}_\epsilon[G](g)$ with respect to the (right) winding automorphisms. Let $\mathfrak{S}(w_1, w_2) = \{1 \leq i \leq r : w_0w_1, w_0w_2 \in \mathrm{Stab}_W(\varpi_i)\}$. Then*

$$N(w_1, w_2) = Q_\ell^w \cap (\sum_{i \notin \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\alpha_i}).$$

Proof. By Theorem 2.8 different blocks arise from the different maximal ideals of Z lying over $\mathfrak{m}_g$, so we need to see how these are permuted by the right winding automorphisms. By 2.4 such maximal ideals are determined by the central elements $b_{\varpi_i}^k c_{\varpi_i}^{\ell-k}$ for $1 \leq i \leq r$, $0 \leq k \leq \ell$. Hence we need only study the action of the winding automorphisms on b_{ϖ_i} and c_{ϖ_i}. By definition $b_{\varpi_i} = c^{\varpi_i}_{f_{-w_0\varpi_i}, v_{\varpi_i}}$ where both $f_{-w_0\varpi_i}$ and v_{ϖ_i} are highest weight vectors. Therefore

$$\Delta(b_{\varpi_i}) = \sum_j c^{\varpi_i}_{f_{-w_0\varpi_i}, v_j} \otimes c^{\varpi_i}_{f_j, v_{\varpi_i}}$$

where $\{f_j\}$ and $\{v_j\}$ are dual bases of $V(\varpi_i)^*$ and $V(\varpi_i)$ respectively. Let $\beta \in Q_\ell$. We have

$$\beta(c^{\varpi_i}_{f_{-w_0\varpi_i}, v_j}) = \begin{cases} \epsilon^{(\beta,\varpi_i)} & \text{if } f_{-w_0\varpi_i} \text{ and } v_j \text{ are dual,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, letting τ_β denote the right winding automorphism of $\mathcal{O}_\epsilon[G]$ defined analogously to those for $U_\epsilon^{\geq 0}$ in 2.11, $\tau_\beta(b_{\varpi_i}) = \epsilon^{(\beta,\varpi_i)} b_{\varpi_i}$. Similarly, we find that $\tau_\beta(c_{\varpi_i}) = \epsilon^{-(\beta,\varpi_i)} c_{\varpi_i}$. We have shown that

$$\tau_\beta(b_{\varpi_i}^k c_{\varpi_i}^{\ell-k}) = \epsilon^{2k(\beta,\varpi_i)} b_{\varpi_i}^k c_{\varpi_i}^{\ell-k}.$$

As in the proof of Theorem 3.3 the maximal ideals of Z lying over $\mathfrak{m}_g$ are obtained by piecing together the maximal ideals of the algebras R_i for $1 \le i \le r$ (notation of the proof of Theorem 3.3). These maximal ideals in turn depend on the vanishing behaviour of $b^\ell_{\varpi_i}(g)$ and $c^\ell_{\varpi_i}(g)$. Specifically if $b^\ell_{\varpi_i}(g) = 0$ or $c^\ell_{\varpi_i}(g) = 0$ then R_i has a unique maximal ideal whilst if $b^\ell_{\varpi_i}(g) \neq 0 \neq c^\ell_{\varpi_i}(g)$ then R_i is semisimple with exactly ℓ maximal ideals. In the second case it follows from the previous paragraph that the winding automorphisms permute the primitive idempotents of R_i non-trivially unless $(\beta, \varpi_i) \in \ell\mathbb{Z}$. Hence the same is true of the maximal ideals. We deduce that the normaliser in Q_ℓ of a block of $\mathcal{O}_\epsilon[G](g)$ is simply the subgroup

$$\{\beta \in Q_\ell : (\beta, \varpi_i) \in \ell\mathbb{Z} \textit{ whenever } b^\ell_{\varpi_i}(g) \neq 0 \neq c^\ell_{\varpi_i}(g)\}.$$

By the proof of Lemma 7.2 this equals the subgroup

$$\{\beta \in Q_\ell : (\beta, \varpi_i) \in \ell\mathbb{Z} \textit{ whenever } w_0w_1, w_0w_2 \in \mathrm{Stab}_W(\varpi_i)\}.$$

The theorem follows from Lemma 2.12(iii). □

7.4. Let $g \in X_{w_1,w_2}$ and let $w = w_2^{-1}w_1$. The factor group $Q^w_\ell / N(w_1, w_2)$ acts simply transitively on the blocks of $\mathcal{O}_\epsilon[G](g)$. We claim that this factor group is an elementary abelian ℓ-group of rank the cardinality of $\mathfrak{S}(w_1, w_2)$, where $\mathfrak{S}(w_1, w_2)$ is as in the statement of Theorem 7.3. To see this recall from the proof of Lemma 2.12 that we have a $< w >$-invariant decomposition $P_\ell = P^w_\ell \oplus P'_\ell$. Moreover, since ℓ is prime to the order of w, we have $Q^w_\ell = (P'_\ell)^\perp$. By definition there is an isomorphism

$$\frac{Q^w_\ell}{N(w_1, w_2)} \cong \frac{Q^w_\ell + \sum_{i \notin \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\alpha_i}}{\sum_{i \notin \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\alpha_i}}.$$

If $i \in \mathfrak{S}(w_1, w_2)$ then $w\varpi_i = (w_0w_2)^{-1}(w_0w_1)\varpi_i = \varpi_i$. Thus $\sum_{i \in \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\varpi_i} \subseteq P^w_\ell$ which in turn implies that $\sum_{i \in \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\varpi_i} \cap P'_\ell = 0$. We deduce that

$$(Q^w_\ell + \sum_{i \notin \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\alpha_i})^\perp = (Q^w_\ell)^\perp \cap (\sum_{i \notin \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\alpha_i})^\perp = P'_\ell \cap \sum_{i \in \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\varpi_i} = 0.$$

Therefore $Q^w_\ell + \sum_{i \notin \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\alpha_i} = Q_\ell$ and we see that the factor group is isomorphic to

$$\frac{Q_\ell}{\sum_{i \notin \mathfrak{S}(w_1,w_2)} \overline{\mathbb{Z}\alpha_i}},$$

an elementary abelian ℓ-group of the required rank. Summing up, we have shown:

Corollary. *Let $g \in X_{w_1,w_2}$. Write $w = w_2^{-1}w_1$.*

1. $\mathcal{O}_\epsilon[G](g)$ has $\ell^{\mathrm{card}(\mathfrak{S}(w_1,w_2))}$ blocks, each containing $\ell^{r-s(w)-\mathrm{card}(\mathfrak{S}(w_1,w_2))}$ simple modules.

2. The quiver of each block of $\mathcal{O}_\epsilon[G](g)$ is a multiply-edged Cayley graph of $N(w_1, w_2)$.

7.5. **Examples.** (i) Suppose first that $g \in X_{e,e}$. Here we have $\mathfrak{S}(e,e) = \emptyset$ and so $N(e,e) = Q_\ell$. In particular there is only one block. The quiver (and the relations) are described in [19].

(ii) Suppose that $g \in G$ lies on the Azumaya locus. Recall from (2.5) that this can be expressed by the condition that $g \in X_{w_1,w_2}$ where $\ell(w_1) + \ell(w_2) + s(w_2^{-1}w_1) = 2N$. Let $u = w_0w_1$, $v = w_0w_2$ and $w = v^{-1}u = w_2^{-1}w_1$. Thus $\ell(u) + \ell(v) = s(w)$. We have inequalities

$$\ell(w) \leq \ell(v^{-1}) + \ell(u) = s(w) \leq \ell(w),$$

which must be equalities. In particular $\ell(w) = s(w)$ so by Lemma 6.8 $w = s_{i_1} \dots s_{i_t}$ where $i_j \neq i_k$ for all $j \neq k$. Since $\ell(v) + \ell(u) = \ell(w)$ we can assume without loss of generality that $v = s_{i_k} \dots s_{i_1}$ and $u = s_{i_{k+1}} \dots s_{i_t}$. It is now clear that

$$\{1 \leq i \leq r : w_0w_1 \in \mathrm{Stab}_W(\varpi_i)\} = I \setminus \{i_{k+1}, \dots, i_t\},$$

and that

$$\{1 \leq i \leq r : w_0w_2 \in \mathrm{Stab}_W(\varpi_i)\} = I \setminus \{i_1, \dots, i_k\}.$$

Hence $\mathfrak{S}(w_1,w_2) = I \setminus \{i_1, \dots, i_t\}$ has cardinality $r - t = r - s(w)$. Thus we have a group of order $\ell^{r-s(w)}$ permuting the blocks simply transitively and a unique simple in each block. This agrees with Theorem 3.3.

(iii) Let $w_1 = w_0s_i = w_2$. In this case $\mathfrak{S}(w_1,w_2) = I \setminus \{i\}$. Thus $\mathcal{O}_\epsilon[G](g)$ has ℓ^{r-1} blocks and each contains $\ell^{r-(r-1)} = \ell$ simple modules.

(iv) Let $G = SL_4(\mathbb{C})$. We will compare two situations:

(a) $w_0w_1 = s_1s_2s_3$, $w_0w_2 = s_1$: here $\mathfrak{S}(w_1,w_2) = \emptyset$;

(b) $w_0w_1 = s_1s_2s_1$, $w_0w_2 = s_1$: here $\mathfrak{S}(w_1,w_2) = \{3\}$.

In both cases we have $\ell(w_1) = 7$, $\ell(w_2) = 9$ and $s(w_2^{-1}w_1) = 2$. This shows that $\ell(w_1), \ell(w_2)$ and $s(w_2^{-1}w_1)$ are not a complete set of invariants for the algebras $\mathcal{O}_\epsilon[G](g)$. This was unknown before!

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